THE

# THEORY, PRACTICE, AND ARCHITECTURE

OF

# BRIDGES

## OF STONE, IRON, TIMBER, AND WIRE;

WITH

## EXAMPLES ON THE PRINCIPLE OF SUSPENSION:

ILLUSTRATED BY

One Hundred and Thirty-eight Engravings

AND NINETY-TWO WOOD-CUTS.

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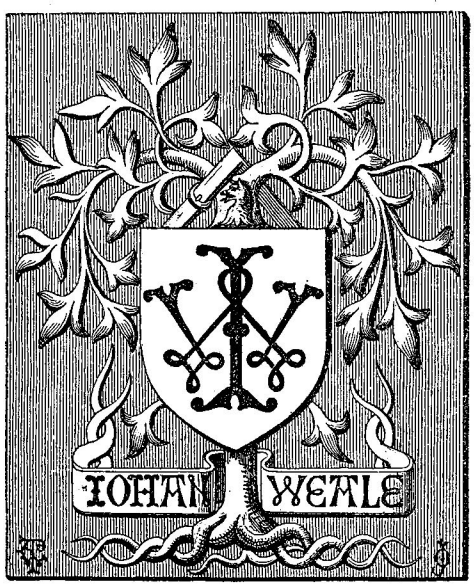
VOL. I.

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# BRIDGES.

## VOL. I.

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THEORY OF BRIDGES,

BY JAMES HANN, KING'S COLLEGE, LONDON.

GENERAL PRINCIPLES OF CONSTRUCTION, &c.

TRANSLATED FROM GAUTHEY.

THEORY OF THE ARCH, &c.

BY PROFESSOR MOSELEY, KING'S COLLEGE.

PAPERS ON FOUNDATIONS,

BY T. HUGHES, C.E.

ACCOUNT OF HUTCHESON BRIDGE, GLASGOW.

SUPPLEMENT.

MATHEMATICAL PRINCIPLES OF DREDGE'S SUSPENSION BRIDGE.

PRINTED BY W. HUGHES,
KING'S HEAD COURT, GOUGH SQUARE.

# ADVERTISEMENT.

THIS work, commenced in 1839, and continued at intervals which have probably exhausted the patience of many of the Subscribers, is at length completed,—the object at first contemplated having been attained by a concentration of talent on this important branch of science, and by the production, it is hoped, of a work on the ART OF BRIDGE BUILDING which will supply the vacant space in the library of the Practical Engineer.

Since the time of Hutton, Attwood, &c., the Theory of the Arch has been deeply investigated in this and other countries; and as the principles of Coulomb and others have caused such investigations to progress extensively, the production of a work on this important subject, combining numerous Theoretical and Practical Examples down to the present period of improvement in the art of construction, necessarily required no inconsiderable portion of time, labour, and expense.

The PRACTICAL TREATISE on BRIDGE BUILDING will, it is anticipated, together with the Specifications and the Paper on FOUNDATIONS, supply all that is desirable for the Student in the pursuit of his knowledge of the Art; while the great variety of engraved specimens by which the work is illustrated, and the

numerous exemplifications of comparative construction, together with the lucid and practical description of them which is appended, will prove interesting to the Engineering public, —and thus compensate for the unavoidable delay which has occurred.

The Analytical List of Contents will guide the reader to the several divisions of the work; and the General Index which I have supplied will be found useful for all the objects of reference.

JOHN WEALE.

59, High Holborn,
Feb. 14, 1843.

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|---|---|---|---|
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| „ | 6 „ | as low . . . . . | as high |
| „ | 5 „ | as high . . . . . | as low |
| 108 | 10 from top | on account of the amount | on account of the smaller amount |
| 138 | 2 „ | minimum . . . . . | maximum |
| 141 | 13 „ | not in words . . . . | not in the same words |
| 149 | 6 from bottom | have piers equal . . . | are equal |
| 160 | 13 from top | faced . . . . . | face |

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Making One Hundred and Thirty-eight Plates,
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# THEORY OF BRIDGES.

THIS very important subject has exercised the talents and ingenuity of some of the greatest mathematicians in modern times, and many different solutions have been given to the various problems connected with it; but, as the greater part of them are founded on suppositions that have no existence whatever either in nature or practice, they have had a tendency rather to mislead than direct those who are engaged in the operations of Bridge Building.

Since the time when Lord Bacon overthrew the absurdities of Aristotle, and showed to the world that experience was the only true guide to philosophy, it might have been expected that theory and practice would have gone hand in hand,—but this unfortunately has not been the case; for we find that their respective advocates have been continually cavilling with each other.

It is to be regretted, deeply regretted, that theoretical and practical men should have always been thus opposed, and have looked upon each other's efforts rather with contempt than admiration, though they seem evidently to have been designed for each other's mutual aid;

and nothing but a deep-rooted prejudice could have continued a system of opposition so destructive of the best interests of society. Our illustrious countryman, Dr. Olinthus Gregory, in the preface to his excellent work on Mechanics, states with his usual elegance, that "theoretical and practical men will most effectually promote their mutual interests, not by affecting to despise each other, but by blending their efforts; and further, that an essential service will be done to mechanical science, by endeavouring to make all the scattered rays of light they have separately thrown upon this region of human knowledge converge to one point."

That a theory may be properly tested, too many facts cannot be collected, too many energies cannot be exerted; for however beautiful may be the theory as far as abstract science is concerned, and however legitimately may the consequences flow from the premises, yet if these premises are not in strict accordance with what is known to take place in actual practice, such theory must ultimately be abandoned, and give place to that which is so founded as to agree with the results of experience and observation. Gauthey, speaking of the theory of La Hire, observes that such analytical researches are unfortunately founded on hypotheses which every day's experience contradicts.

We shall in the first place give a brief account of the principal writers on the equilibrium of the arch, with some notice of their theories.

In 1691, the celebrated mathematicians, Leibnitz, Huygens, James and John Bernoulli, solved the problem

of the catenary curve: it was soon preceived that this was precisely the curve that should be given to an arch, the materials of which were infinitely small and of equal weight, in order that all its parts may be in equilibrium. In the Philosophical Transactions for the year 1697, it appears that David Gregory first noticed this identity, but his mode of argument, though sufficiently rigorous, appears not to be so perspicuous as could be desired.

In one of the posthumous works of James Bernoulli, two direct solutions of this problem are given, founded on the different modes of viewing the action of the voussoirs: the first is clear, simple, and precise, and easily leads to the equation of the curve, which he shows to be the catenary inverted; the second requires a little correction, which Cramer, the editor of his works, has pointed out.

In 1695, La Hire,[1] in his Treatise on Mechanics, laid down from the theory of the wedge, the proportion according to which the absolute weight of the materials of masonry ought to be increased from the key-stone to the springing in a semicircular arch. The historian of the Academy of Sciences relates in the volume for the year 1704, that Parent determined on the same principle, but only by points, the figure of the extrados of an arch, the intrados being a semicircle, and found the force or thrust of a similar arch against the piers.

In the memoirs of the Academy of Sciences for the year 1712, La Hire gave an investigation of the thrusts

[1] Mr. Attwood has written a dissertation on the construction of arches on the same principles as La Hire.

in arches under a point of view suggested by his own experiments: he supposed that the arches, the piers of which had not solidity enough to resist the thrust, split towards the haunches at an elevation of about 45 degrees above the springings or impost; he consequently regarded the upper part of the arch as a wedge that tends to separate or overturn the abutments, and determined, on the theory of the wedge and the lever, the dimensions which they ought to have to resist this single effort.

Couplet, in a memoir composed of two parts, the first of which was printed in the volume of the Academy for 1729, treats of the thrusts of arches and the thickness of the voussoirs, by considering the materials infinitely small, and capable of sliding over each other without any pressure or friction. But, as this hypothesis is not exactly conformable to experiment, the second part of the memoir, printed in the volume for 1730, resumes the question by supposing that the materials have not the power of sliding over each other, but that they can raise themselves and separate by minute rotatory motions. It cannot however be said that Couplet has added materially to the theories of La Hire and Parent, and none of them treated the subject either in theory or practice in such a satisfactory manner as was afterwards done by Coulomb.

In a subsequent volume there is a memoir by Bouguer on the curve lines that are most proper for the formation of the arches of domes. He considers that there may be an infinite number of curve lines employed for this purpose, and points out the mode

of selecting them. He lays it down uniformly that the voussoirs have their surfaces infinitely smooth, and establishes, on this hypothesis, the conditions of equilibrium in each horizontal course of the dome, but has not given any method of investigating the thrusts of arches of this kind, nor of the forces that act upon the mason-work when the generating curve is subjected to given conditions.

In 1770, Bossut gave investigations of arches of the different kinds, in two memoirs, which were printed among those of the Academy of Sciences for the years 1774 and 1776: he appears to have been engaged in this in consequence of some disputes concerning the dome of the church of St. Genevieve (recently the French Pantheon), begun by the celebrated architect Soufflot, and finished from his designs.

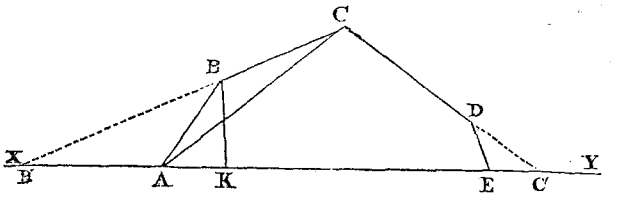
In 1772, Dr. Hutton, late Professor of Mathematics in the Royal Military Academy, Woolwich, published his principles of Bridges, in which he investigated the form of curves for the intrados of an arch, the extrados being given, and *vice versa*. He set out by developing the properties of the equilibrated polygon, which is extremely useful in the equilibrium of structures.

## ON THE EQUILIBRATED POLYGON.

Any number $n$ of bars or beams A B, B C, C D, &c., of given weights $\omega_1$, $\omega_2$, $\omega_3$, &c. . . . . $\omega_n$, and supporting given weights $W_1$, $W_2$, $W_3$, &c., at the respective points B, C, D, &c., are freely moveable about

each other in the same vertical plane A B C D, &c., by means of joints at those points, and are supported at the extremities of the first and last, and acted on by gravity. It is required to investigate expressions for determining their respective positions, when in a state of equilibrium.

Considering the two bars AB and BC as fixed at their extremities A and C, their positions are determined as they can form but one triangle ABC upon the constant side A C. But




supposing the three bars A B, B C, and C D, as supported at A and D, their positions become indeterminate, that is, they are moveable about those points, and an equation is necessary to find their respective positions when in a state of equilibrium. We shall, in the first place, take the three bars A B, B C, and C D.

Produce C B, C D, &c., to meet the horizontal line X A Y at B′, C′, &c., draw B K perpendicular to this line, and let B A Y, the elevation of A B to the horizon $= \theta_1$, B B′ Y; that of B C $= \theta_2$; C C′ Y, the elevation of C D $= \theta_3$, &c.

Suppose the weight $W_1$ to be fixed to the end of AB, and let the vertical pressure acting at B upon A B in direction B K, and upon B C in direction K B $= p$, the lateral force acting at B upon A B in direction K A, and upon B C in direction A K, being equal to that acting at C upon B C in direction K A $= f$.

Then the forces $p$ and $f$ united with gravity having no effect to turn AB about A, we have the equation

$$\left(\frac{\omega_1}{2}+\mathrm{W}_1\right)\cos.\,\theta_1+p\cos.\,\theta_1-f\sin.\,\theta_1=0,$$

$$\therefore\frac{\omega_1}{2}+\mathrm{W}_1+p-f\tan.\,\theta_1=0.$$

And since the tendency of the same forces at B, combined with gravity, to turn BC about the joint C, is destroyed in the position of equilibrium, we have also

$$\frac{\omega_2}{2}\cos.\,\theta_2-p\cos.\,\theta_2+f\sin.\,\theta_2=0,$$

$$\therefore\frac{\omega_2}{2}-p+f\tan.\,\theta_2=0.$$

To this equation add the former, and we have

$$\frac{\omega_1+\omega_2}{2}+\mathrm{W}_1-f\,(\tan.\,\theta_1-\tan.\,\theta_2)=0,$$

$$\therefore f=\frac{\frac{1}{2}(\omega_1+\omega_2)+\mathrm{W}_1}{\tan.\,\theta_1-\tan.\,\theta_2}.$$

Now since the centre of gravity of BC is at rest, the lateral forces at B and C in opposite directions must be equal

$$\therefore f=\frac{\frac{1}{2}\,(\omega_1+\omega_2)+\mathrm{W}_1}{\tan.\,\theta_1-\tan.\,\theta_2}=\frac{\frac{1}{2}\,(\omega_2+\omega_3)+\mathrm{W}_2}{\tan.\,\theta_2-\tan.\,\theta_3}\ .\ .\ .\ .$$

$$=\frac{\frac{1}{2}\,(\omega_{n-1}+\omega_n)+\mathrm{W}_{n-1}}{\tan.\,\theta_{n-1}-\tan.\,\theta_n}\quad .\ .\ .\ .\ .\ .\ .\ .\ .\quad (\mathrm{F})$$

consequently

$$\tan.\,\theta_1-\tan.\,\theta_2=\frac{1}{f}\left(\frac{\omega_1+\omega_2}{2}+\mathrm{W}_1\right)\ .\ .\ .\quad (1)$$

$$\tan.\,\theta_2-\tan.\,\theta_3=\frac{1}{f}\left(\frac{\omega_2+\omega_3}{2}+\mathrm{W}_2\right)\ .\ .\ .\quad (2)$$

$$\text{tan. } \theta_3 - \text{tan. } \theta_4 = \frac{1}{f}\left(\frac{\omega_3+\omega_4}{2} + W_3\right) \quad . \quad . \quad . \quad (3)$$

&c. &c. &c. &c.

$$\text{and tan. } \theta_{n-1} - \text{tan. } \theta_n = \frac{1}{f}\left(\frac{\omega_{n-1}+\omega_n}{2} + W_{n-1}\right) \ldots . (n-1)$$

By means of these $n-1$ equations involving $n+1$ unknown quantities, *viz.*, tan. $\theta_1$, tan. $\theta_2$ . . . tan. $\theta_n$ and $f$, the positions of any number of bars may be obtained so as to render them in a state of equilibrium, having previously given two more data, *viz.*, the positions of any two of them: thus suppose $\theta_p$ and $\theta_q$ the elevations of the $p^{\text{th}}$ and $q^{\text{th}}$ bars to be given, then if we begin at equation $(p)$, and add all the equations up to the $(q-1)^{\text{th}}$ we have

$$\text{tan. } \theta_p - \text{tan. } \theta_q = \frac{1}{f}\left\{(\omega_p+\omega_{p+1} \ldots \omega_q) - \frac{\omega_p+\omega_q}{2} + (W_p+W_{p+1} \ldots +W_{q-1})\right\},$$

$$\therefore f = \frac{(\omega_p+\omega_{p+1}..\omega_q) - \frac{1}{2}(\omega_p+\omega_q) + (W_p+W_{p+1}..W_{q-1})^*}{\text{tan. } \theta_p - \text{tan. } \theta_q}.$$

If in equation (F) we suppose the weights of the bars $=0$, we have

$$\frac{W_1}{\text{tan. } \theta_1 - \text{tan. } \theta_2} = \frac{W_2}{\text{tan. } \theta_2 - \text{tan. } \theta_3},$$

$$\therefore W_1 : W_2 :: \text{tan. } \theta_1 - \text{tan. } \theta_2 : \text{tan. } \theta_2 - \text{tan. } \theta_3.$$

Hence the weights $W_1$, $W_2$, &c., are as the difference of the tangents of the angles which the bars make with the horizon, the same as the proportion of the

* For the equilibrium of any number of bars without weights, an investigation similar to the above has been given by that able mathematician, Mr. Woolhouse, in the Newcastle Magazine.

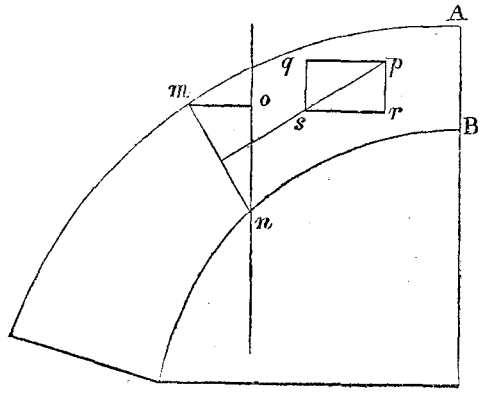
weights on the funicular polygon. See Dr. Gregory's Mechanics, page 134; Whewell's Mechanics, page 34.

If $W_1, W_2$, &c.$=0$; then $\frac{\frac{1}{2}(\omega_1+\omega_2)}{\tan.\theta_1-\tan.\theta_2}=\frac{\frac{1}{2}(\omega_2+\omega_3)}{\tan.\theta_2-\tan.\theta_3}$.

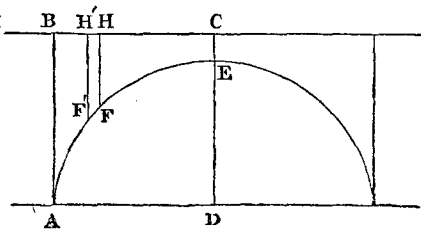
When the weights of the bars are also equal, we have $\tan.\theta_1-\tan.\theta_2=\tan.\theta_2-\tan.\theta_3$, &c...$=\tan.\theta_{n-1}-\tan.\theta_n$. Since the weights on the several joints are as the difference of the tangents of the angles of elevation, when these bars are indefinitely diminished, or the polygon becomes the curve of equilibration, we have ultimately the weight on any point proportional to the tangent of inclination to the horizon.[1]

[1] Or this may be shown from the principles of Coulomb, Mem. Pres. &c., tom. vii.: he there shows that if an arch rest in equilibrium on a base *mn*, it is necessary that the resultant of the two acting forces, viz., the horizontal force acting at AB, and the weight of the part A *mn* B acting vertically, shall be perpendicular to *mn*, and that it shall not fall without *mn*. Should the first condition be wanting, the arch will yield by sliding along *mn*; and if the second be wanting, it will have a rotatory motion about that extremity towards which the resultant falls.

Let W be the weight of the portion A *mn* B, F the horizontal force on AB, $\iota$ the inclination of the joint *mn* to the vertical; then if the weight of the part A *mn* B be represented by the vertical line *pr*, and the force F by the horizontal line *pq*, which meet in *p*, we have, since the resultant *ps* is perpendicular to *mn*, the triangles *psq* and *mon* similar;



From this property we may immediately derive one of the most useful cases of equilibration, viz. The extrados being a horizontal line, it is required to determine the intrados, on the supposition that the equilibrium is maintained by vertical pressures on the voussoirs. Let C be the origin of co-ordinates, CH$=x$, HF$=y$, CE$=a$; CD$=\alpha$, AD$=\beta$, $\iota$ the inclination of the tangent to the horizon, and draw H′F′ indefinitely near to HF. Now if we suppose the whole mass to act perpendicularly on the voussoirs, the differential of the weight of the column may be represented by HF. HH′$=y\,dx$, and since $\frac{dy}{dx}$ =tang. of inclination, we have from the above $f\,y\,d\,x = c\,\frac{dy}{dx}$ or $y\,dx = c.\ d.\frac{dy}{dx}$; $c$ being a constant quantity, which will be determined afterwards:




$$\therefore y\,dx^2 = c\,d^2y.$$

Multiplying by $dy$, and integrating, we have

$$\tfrac{1}{2}\,(y^2 + \text{const.})\,dx^2 = \tfrac{1}{2}\,c\,dy^2$$

---

$\therefore pr : rs\ (pq) :: mo : on$

W : F :: sin. $\iota$ : cos. $\iota$ :: tan. $\iota$ : 1,

$\therefore$ W=F. tan. $\iota$, consequently W varies as tan. $\iota$, since the horizontal force is constant.

Also since $mn$ is a normal, and tan. $\iota=\frac{dy}{dx}$, we have

$$\text{W} = \text{F}.\ \frac{dy}{dx} \text{ or } \text{W}\,dx = \text{F}\,dy,$$

an equation to the catenary.

$$\text{or } c\left(\frac{dy}{dx}\right)^2 = y^2 + \text{const.}$$

But when $y = a$, $\frac{dy}{dx}$ vanishes,

$$\therefore \text{the correct integral is } c\left(\frac{dy}{dx}\right)^2 = y^2 - a^2$$

$$\therefore \frac{dy}{dx} = \frac{\sqrt{y^2 - a^2}}{\sqrt{c}}, \text{ or } dx = \sqrt{c}.\frac{dy}{\sqrt{y^2 - a^2}},$$

the correct integral of which, Hall's Diff. and Int. Calculus, page 313, is

$$x = \sqrt{c} \cdot \left(\text{hyp. log. } \frac{y + \sqrt{y^2 - a^2}}{a}\right)$$

$$\therefore \frac{x}{\sqrt{c}} = \text{hyp. log. } \frac{y + \sqrt{y^2 - a^2}}{a}.$$

If $e$ be the hyperbolic base, we have

$$e^{\frac{x}{\sqrt{c}}} = \frac{y + \sqrt{y^2 - a^2}}{a}$$

$$e^{-\frac{x}{\sqrt{c}}} = \frac{a}{y + \sqrt{y^2 - a^2}} = \frac{y - \sqrt{y^2 - a^2}}{a}.$$

By addition we have

$$e^{\frac{x}{\sqrt{c}}} + e^{-\frac{x}{\sqrt{c}}} = \frac{2y}{a}$$

$$y = \frac{a}{2}\left(e^{\frac{x}{\sqrt{c}}} + e^{-\frac{x}{\sqrt{c}}}\right).$$

Differentiating,

$$\frac{dy}{dx} = \frac{a}{2\sqrt{c}}\left(e^{\frac{x}{\sqrt{c}}} - e^{-\frac{x}{\sqrt{c}}}\right),$$

which shows that the curve is convex towards the axis of $x$.

To find the constant $c$, we must observe that when F comes to A, $y = \mathrm{C\,D} = \alpha$, and $x = \mathrm{A\,D} = \beta$,

$$\therefore \beta = \sqrt{c}.\ \text{hyp. log.}\ \frac{\alpha + \sqrt{\alpha^2 - a^2}}{a}$$

$$\text{or } \sqrt{c} = \beta \div \text{hyp. log.}\ \frac{\alpha + \sqrt{\alpha^2 - a^2}}{a}.$$

Hence by substitution we have

$$x = \beta.\ \text{hyp. log.}\ \frac{y + \sqrt{y^2 - a^2}}{a} \div \text{hyp. log.}\ \frac{\alpha + \sqrt{\alpha^2 - a^2}}{a}.$$

Also, from the above, another property may be derived, on which Dr. Hutton bases many of his calculations. If $\rho$ = radius of curvature, then we have

$$\rho = \frac{dz^3}{dy\,d^2x - dx\,d^2y}, \text{ which becomes,}$$

if we suppose $x$ to be the independent variable,

$$\rho = \frac{dz^3}{dx\,d^2y},$$

since the curve is convex to the axis of $x$,

and
$$c\,d.\frac{dy}{dx} = y\,dx$$

$$\therefore \frac{d^2y}{dx^2} = \frac{y}{c},$$

$$\text{but } dz = \sqrt{dx^2 + dy^2} = dx\sqrt{1 + \frac{dy^2}{dx^2}};$$

hence by substitution

$$\rho = \frac{c\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{y},$$

$$\text{but } \frac{dy}{dx} = \tan.\iota,$$

$$\therefore \rho = \frac{c \sec.^3\iota}{y}.$$

Consequently the radius of curvature is proportional to $\sec.^3\iota$ directly, and HF inversely, or HF is as $\sec.^3\iota$ directly, and the radius of curvature inversely, $c$ being constant.

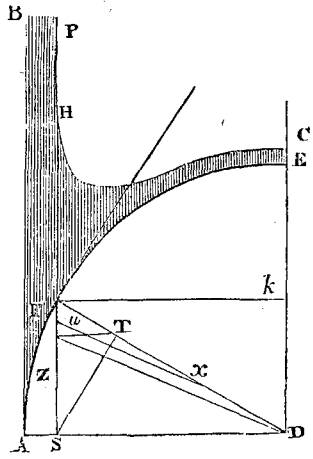
At the vertex E of any curve the inclination is nothing; therefore $\sec.\iota = 1$, and if the radius of curvature there be represented by $\rho'$, the general expression for the height becomes $a = \frac{c}{\rho'}$ or $c = a\,\rho'$, which is the general value of $c$ for any curve, in terms of the height of the crown and radius of curvature at that point: this substituted in the general expression, we have

$$\text{HF} = \frac{c}{\rho} \sec.^3\iota = \frac{a\rho'}{\rho} . \sec.^3\iota.$$

If the arc be the segment of a circle, then $\frac{\rho'}{\rho} = \frac{\rho}{\rho} = 1$,

$$\therefore \text{HF} = a \sec.^3\iota.$$

This may be very simply calculated by logarithms, log. HF = log. $a$ + 3 log. sec. $\iota$. Or we may give a geometrical construction as follows:




Draw the vertical line FS cutting the horizontal diameter in S, draw ST perpendicular to the radius DF, draw the horizontal line T$z$, cutting the vertical in $z$, join D$z$. Make F$u$ = CE, and draw $ux$ parallel to

$z$D; then FH must be equal to F$x$, and by similar triangles,

$$\text{FD} : \text{FS} : : \text{FS} : \text{FT} = \frac{\text{FS}^2}{\text{FD}},$$

$$\text{FS} : \text{FT} : : \text{FT} : \text{F}z = \frac{\text{FS}^4}{\text{FD}^2 . \text{FS}} = \frac{\text{FS}^3}{\text{FD}^2},$$

$$\text{CE} : \text{FH} : : \text{F}u : \text{F}x : : \text{F}z : \text{FD} : : \frac{\text{FS}^3}{\text{FD}^2} : \text{FD} : : \text{FS}^3 : \text{FD}^3 : : \text{D}k^3 : \text{DE}^3.*$$

The curve CHP runs up to an infinite height above the spring of the arch, and this must evidently be the case with every curve that springs at right angles to the horizontal line.

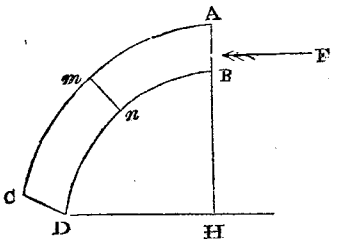
From the above it appears, that a semicircular arch cannot be put in equilibrio by building upon it, whatever may be its span or thickness at the crown; since the curve CHP runs up indefinitely, having AB for its assymptote; and therefore, according to the principles of equilibration, it is not adapted for a bridge which requires an outline nearly horizontal, except for about 30 or 40 degrees on each side of the vertex C.

From the general expression $\text{HF} = \frac{c}{\rho} \sec.^3 \iota$, and the equation of the curve, we may find either the extrados or intrados, whichever may be the given curve.

The celebrated experimental philosopher Coulomb,

* Dr. Gregory's "Mechanics," art. Arches.

to whom practical science is so deeply indebted, seems to have been the first who treated this important subject in a manner conformable to experiment and observation: he considers the equilibrium of arches successively on the hypothesis of joints perfectly polished, and on that where the friction of mortars or cements is taken into consideration. In the first place, he takes the part AB*nm* of the half arch ABCD as a body supported on an inclined plane, and gives, having respect to the effects of friction and cohesion, the limits between which the force F acting horizontally at some point in the joint AB must fall, so that the arch may be prevented from sliding along the joint *m n*, either downwards in direction *m n*, or upwards in direction *nm*, or by rotation inwards round the point *n* of the joint *m n*, or outwards round the point *m*. The general expression for F being determined, he finds the maximum value of it, which will prevent the arch from sliding in direction *m n*, and the minimum value, which will cause it to slide in the opposite direction *n m*: he then observes that the limit of F, or the force which will be just sufficient to preserve the equilibrium, must be greater than the maximum and less than the minimum values so obtained.



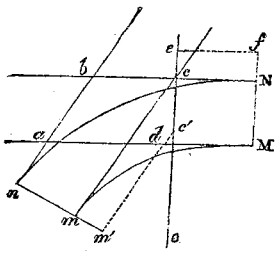

He then considers what the conditions of equilibrium must be, so that rotation cannot take place either round *m* or *n*, and finds in a similar way the maximum value of F, which will prevent rotation round the point *n*, and the minimum value that will cause it to move round *m*,

and shows that to maintain the equilibrium the force must be greater than the maximum and less than the minimum.

From this we have two superior and two inferior limits obtained for the force F, between which it must be found, so that the arch cannot slide either in the direction $m\,n$ or $n\,m$, nor yet revolve round either of the points $m$ or $n$. Gauthey, speaking of the analysis of Coulomb, observes that it leaves nothing wanting to make it coincide with that to which we are conducted by the latest experiments on the strength of arches.

That distinguished individual, the Rev. H. Moseley, Professor of Natural Philosophy in King's College, London, has, in two excellent papers in the Transactions of the Cambridge Philosophical Society, developed with great ability a theory for the equilibrium of the arch, on principles somewhat different from those of Coulomb, and has deduced expressions which accord, not only with a great many experiments made by himself, but also with those of Gauthey and Professor Robison. As this learned gentleman has kindly consented to give us some account of this theory, which appears to include that of Coulomb, we shall, for the present, defer entering into his views on this subject, considering that he will be better able to explain them himself; but we may observe, that it is our intention, in this work, to enter fully into the merits of both, and give as clear an exposition of each as we possibly can. Professor Moseley's Paper will be found in a future portion of this work.

We have, in the preceding pages, given a general outline of the celebrated theory of equilibration, which has long been considered by many eminent writers as one of the most delicate researches in applied mathematics, and indeed it seems to be almost too delicate for practical purposes, being based on suppositions which cannot hold good in practice; neither is the equilibrium sufficiently stable except under certain circumstances; and to ensure the stability of this equilibrium some writers have shown that the vertical line through the centre of gravity of the part *mn*NM should fall within the parallelogram *abcd*, formed by perpendiculars drawn at the extremities of the lines MN, *mn*, for otherwise the resultant of the two forces, viz., the weight of the part *mn*NM, and the horizontal thrust, would fall without *nm* where the arch rests, and from no point of that vertical could there be drawn two perpendiculars, the one to *mn* and the other to MN, so as to fall within both; consequently the part *mn*NM could not be sustained by those two joints or supports.




At the extreme point *c*, if the vertical line *ce* should pass through the centre of gravity, it is evident that from no other point in this line except *c* could there be drawn two perpendiculars, one to *mn*, and the other to MN; for if from any point *c'* below *c*, *c'm'* be drawn, the point *m'* will fall on *nm* produced, and if from any point *e* in this line above *c*, *ef* be drawn, it will fall in MN produced; *c* is therefore the utmost limit from

which the vertical line passing through the centre of gravity can be drawn for the equilibrium to have place, and it is clear that this is but a very unstable equilibrium, for the slightest force would derange it by throwing the vertical line beyond the required limits.

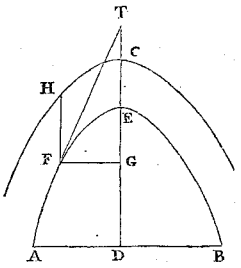
We shall now give a few examples to show how the extrados may be determined when the intrados is the given curve.

*The intrados being a Parabola, required the extrados.*

Let E be the origin, EG$=x$, FG$=y$, then

$$\rho = \frac{(m+4x)^{\frac{3}{2}}}{2\sqrt{m}} \text{ and } \rho' = \tfrac{1}{2}m \text{ and sec. } \iota = \frac{\text{FT}}{\text{FG}}$$

$\rho$ and $\rho'$ being the radii of curvature at F and E as before, and since $y^2 = m\,x$, and $\text{GT} = 2\,x$, we have sec. $\iota = \sqrt{\left(\frac{m+4\,x}{m}\right)}$; and by page 13,




$$\text{HF} = \frac{a\,\rho'}{\rho}\,\text{sec.}^3\iota = a\,\frac{\frac{1}{2}m\,.\,2\,m^{\frac{1}{2}}}{(m+4\,x)^{\frac{3}{2}}}\,.\,\frac{(m+4\,x)^{\frac{3}{2}}}{m^{\frac{3}{2}}}.\; = a.$$

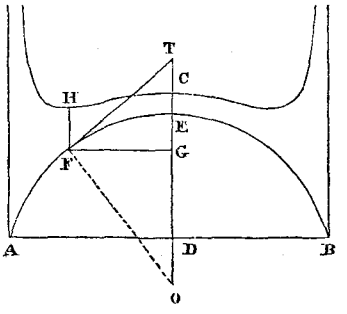
Hence it follows that the extrados is also a parabola equal to the intrados, and every where vertically equidistant from it.

*To find the thickness over any point of an Elliptic arch.*

Retaining the same notation for $x$, $y$, and $a$, let the semi-axis major $= b$, the semi-axis minor $= c$, and F 0$=h$.

$DE^2 : AD^2 :: DG : OG$

$c^2 : b^2 :: c-x : \frac{b^2}{c^2}(c-x) = OG$




$\text{sec. } \iota = \frac{FO}{GO} = \frac{hc^2}{b^2(c-x)}$, $\rho = \frac{4h^3}{m^2}$,

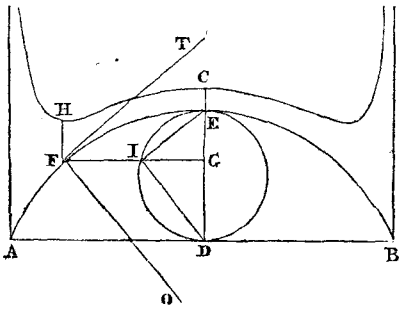
$m$ being the parameter to $DE = \frac{2b^2}{c}$.

$$\therefore \rho = \frac{c^2 h^3}{b^4} \text{ also } \rho' = \frac{b^2}{c}$$

$$HF = \frac{a\rho'}{\rho} \text{sec.}^3 \iota = \frac{ab^4}{c^2 h^3} \cdot \frac{b^2}{c} \cdot \frac{c^6 h^3}{b^6 (c-x)^3} = \frac{ac^3}{(c-x)^3}.$$

*For the Cycloid.*

Let $2r$ be the diameter of the generating circle, then in this curve the following properties obtain, viz., the tangent FT is parallel to the corresponding chord EI, or the angle TFG = the angle EIG, also the radius of curvature at F is parallel to the supplemental chord ID, and equal to twice that chord, or $\rho = 2ID$,




$EI = \sqrt{2rx}$, $GI = \sqrt{2rx - x^2}$, $DI = \sqrt{4r^2 - 2rx}$,

and therefore

$$\rho = 2\sqrt{4r^2 - 2rx}, \text{ and } \rho' = 2DE = 4r$$

$$\text{also sec. } \iota = \frac{EI}{GI} = \sqrt{\left(\frac{2rx}{2rx - x^2}\right)} = \sqrt{\left(\frac{2r}{2r - x}\right)}$$

$$\therefore \text{HF} = \frac{a\rho'}{\rho} \sec.^3\iota = \frac{a \,.\, 4r}{2\sqrt{(4r^2-2rx)}} \cdot \frac{(2r)^{\frac{3}{2}}}{(2r-x)^{\frac{3}{2}}}$$

which by reduction becomes $\frac{4ar^2}{(2r-x)^2}$.

It may here be observed that in both the ellipse and the cycloid, the extrados is analogous to that of the circle, but somewhat flatter, and by computing values for FH the extrados is easily constructed by points.

We have throughout the whole of these articles considered the curve of equilibration to coincide exactly with the intrados, but this does not appear to be the most advantageous position for it to have. Dr. Thomas Young, in one of the ablest written articles which has appeared on this subject (see Napier's edition of the Encyclopædia Britannica), observes that "when the curve of equilibrium touches the intrados of an arch of any kind, the compression at the surface must be at least four times as great as if it remained at the middle of the arch-stones." This perhaps might be too high to fix its position when the bridge is constructed, since it is always found that arches settle either more or less after the centering is struck: the settlement then would throw the curve of equilibration above the middle of the arch-stones at the crown, but no allowance can be made for this, as it cannot be determined *à priori* what any arch will settle. If this curve coincides with the intrados, at the crown, before the centering is struck, it will undoubtedly afterwards take its place within the arch-stones, which is a

more advantageous position, so far as strength or durability of the arch is concerned.

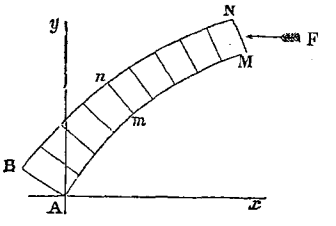
We shall now proceed to take a different view of this subject, and to show the conditions of equilibrium that must exist, in order that the arch may be so tied up that the voussoirs can neither slide upwards nor downwards, nor yet turn round either the upper or lower edges of their joints. This method of establishing the equilibrium of the arch has not been considered by any English writer till very lately, and even on the Continent, the place of its birth, it has not till within late years met with that attention which it so highly merits. The theory of Coulomb, which proceeds on this principle, has scarcely ever been noticed in this country; nay, the only English works we are acquainted with that inform us that Coulomb had at all written on the theory of arches are Cresswell's Venturoli, Dr. Gregory's edition of Dr. Hutton's course, and some of the writings of Professor Moseley.

*On the equilibrium of an assemblage of voussoirs.*

1. Let ABNM represent part of an arch, inclining at AB against a fixed plane, and supported at the other extremity by a force which is usually denominated the horizontal thrust. The form of this part of the arch is given by the curves of intrados and extrados, and the direction of the planes of the joints.

Now if we consider this part of the arch to be formed

by placing voussoirs successively upon each other, beginning at the fixed plane AB, it is clear that the first voussoirs placed against this plane would be supported merely by the effects of friction, and would continue to be so supported until the inclination of the joints became so great as to cause them to slide;[1] it will then be necessary to apply to the joint MN of the last voussoir a force F, whose vertical and horizontal components may be represented by P and Q. This force ought to be sufficient to prevent the voussoirs from sliding downwards on the planes of the joints, and also to prevent them from turning on their lower edges; but it ought not to be so great as to cause the voussoirs to slide upwards on the joints, or to produce a rotary motion round their upper edges. We therefore see




[1] Mr. George Rennie, in a valuable paper in the Philosophical Transactions for 1829, states that "the granite voussoirs of the arches of the New London Bridge, having their beds well faced and dressed without mortar, generally commence sliding at angles from 33° to 34°. But with a bed of fresh and finely ground mortar interposed, the pressure on the centering commences at angles of from 25° to 26°. In other cases of arches, where sand-stones, such as Bramley Fall and Whitby, were employed, and their beds faced and dressed as usual, the angle of sliding was found to vary from 35° to 36°. But with mortar interposed, the angle generally varied from 33° to 34°.

"It results from these and other experiments that friction, by absorbing part of the horizontal thrust, is a most powerful assistant in maintaining the equilibrium of arches, and enables us to determine with something like precision the allowances due to theory."

generally that taking any joint $mn$ whatever, the system of forces applied to the arch ABMN comprised in the components P and Q applied at the upper joint of the last voussoir, ought to be such that the action of the forces applied to the upper part $mn$NM cannot cause that part to slide on the plane of the joint $mn$, nor yet turn round either of the edges $m$ or $n$.

Let $x$ and $y$ be the horizontal and vertical co-ordinates of the point $m$.

$x'$, $y'$, those of the point $n$.

$\theta$ the angle which the joint $mn$ forms with the vertical.

$z$, the length of the joint $mn$.

$a$, $b$, the co-ordinates of the point M of the curve of intrados.

$a'$, $b'$, the co-ordinates of the point N of the curve of extrados.

W, $\omega$, the vertical and horizontal components resulting from the weight of the portion $mn$NM of the arch.

$\alpha$, $\beta$, the co-ordinates of the point $c$, where the components W and $\omega$ act.

$f$, the coefficient of friction, which is supposed to be proportional to the pressure.

$r$, the cohesive force of a unit of surface of the joint to prevent sliding.

R, the cohesive force of a unit of surface, which tends to prevent rotation.

T, the pressure in direction perpendicular to the joint $m\,n$.

2. To investigate the conditions relative to sliding on the joint $mn$, the force which tends to make the portion of the arch M$mn$N slide in the direction $nm$, is

$$(\mathrm{P}+\mathrm{W})\cos.\,\theta$$

and the force which opposes this sliding is

$$(\mathrm{Q}+\omega)\sin.\,\theta+f\,(\mathrm{P}+\mathrm{W})\sin.\,\theta+f\,(\mathrm{Q}+\omega)\cos.\,\theta+rz.$$

Now, that sliding may not take place in the direction $nm$, the latter must be greater than the former, or

$$\mathrm{P}\,(1-f\tan.\,\theta)-\mathrm{Q}\,(f+\tan.\,\theta)<-\mathrm{W}\,(1-f\tan.\,\theta)$$
$$+\omega\,(f+\tan.\,\theta)+\frac{rz}{\cos.\,\theta}.$$

Also the force which tends to make the same portion slide in the opposite direction $mn$ is

$$(\mathrm{Q}+\omega)\sin.\,\theta$$

and the force which opposes this sliding is

$$(\mathrm{P}+\mathrm{W})\cos.\,\theta+f\,(\mathrm{P}+\mathrm{W})\sin.\,\theta+f\,(\mathrm{Q}+\omega)\cos.\,\theta+rz.$$

Hence, that the sliding may not take place in the direction $mn$, we must have

$$-\mathrm{P}\,(1+f\tan.\,\theta)+\mathrm{Q}\,(\tan.\,\theta-f)<\mathrm{W}\,(1+f\tan.\,\theta)$$
$$-\omega\,(\tan.\,\theta-f)+\frac{rz}{\cos.\,\theta}.$$

3. We must now proceed to establish the conditions that rotation may not take place round either of the points $m$ or $n$.

Let us suppose, in the first place, that the portion of the arch $mn$NM tends to turn from the top to the

bottom on the edge or arris $m$, and that the force whose resultants are P and Q is applied at the point N, where it will have the least tendency to produce rotation round the edge $m$.

The moment of the forces which tend to make $mn$NM turn round $m$ is

$$P(a'-x)+W(\alpha-x)$$

and the moment of the forces which oppose this rotation is

$$Q(b'-y)+\omega(\beta-y);$$

but if we take into account the resistance from cohesion, we may adopt the theory of Mariotte and Leibnitz, viz., that the resistance to rupture which cohesion exerts at different points of the joint between $m$ and $n$, will be proportional to the distances of those points from $m$.

Let an element of the joint $mn$ be represented by $dv$, and $v$ its distance from the point $m$, the cohesion of this element will be $Rdv$, and the resistance exerted by this cohesion is $\frac{R}{z}vdv$, the moment of this resistance round the point $m$ will be $\frac{R}{z}v^2dv$; its integral taken between the limits of $v=o$, and $v=z$, or as it is usually expressed,

$$\int_0^z \frac{R}{z}v^2dv=\tfrac{1}{3}Rz^2,$$

the moment from cohesion; hence the above becomes, taking into account the effects of cohesion,

$$Q(b'-y)+\omega(\beta-y)+\tfrac{1}{3}Rz^2;$$

therefore, in order that rotation may not take place round the point $m$, we must have

$$P(a'-x)-Q(b'-y) < -W(a-x)+\omega(\beta-y)+\tfrac{1}{3}Rz^2.$$

Now, in the next place, if we consider the portion $mn$NM tends to turn on the edge $n$, and that the force whose resultants are P and Q is applied at M, where it will have the least tendency to produce this rotation; then the moment of the forces which tend to produce this motion is

$$Q(b-y')+\omega(\beta-y')$$

also the moment of the forces which oppose this motion is

$$P(a-x')+W(\alpha-x')+\tfrac{1}{3}Rz^2.$$

Therefore, in order that this rotation may not take place, we must have

$$-P(a-x')+Q(b-y') < W(\alpha-x')-\omega(\beta-y')+\tfrac{1}{3}Rz^2$$

The equilibrium of the portion $mn$NM requires, besides the condition mentioned in the last article, that the components P and Q should be such that these last conditions will also be satisfied for any joint $mn$.

Conversely, when the preceding conditions are satisfied for all the joints, the arch will necessarily remain in equilibrium.

4. The pressure in direction perpendicular to the joint $mn$, is

$$T=(P+W)\sin.\theta+(Q+\omega)\cos.\theta.$$

5. If in the preceding conditions relative to sliding we suppose the resistance from friction and cohesion to be nothing, we have, for a strict equilibrium,

$$P - Q \tan. \theta = -W + \omega \tan. \theta$$

$$\therefore \tan. \theta = \frac{P+W}{Q+\omega},$$

which shows that the resultant of the forces applied to the portion of the arch $mn$NM must be perpendicular to the joint $mn$.

6. Also the conditions relative to rotation become, when the friction and cohesion are not taken into account,

$$P(a'-x) - Q(b'-y) < -W(\alpha - x) + \omega(\beta - y)$$
$$-P(a-x') + Q(b-y') < W(\alpha - x') + \omega(\beta - y')$$

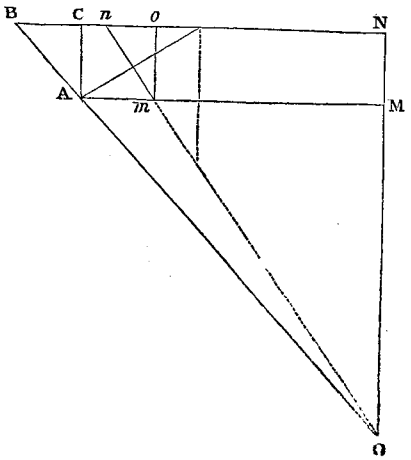
which show that the resultant of the forces applied to $mn$NM must fall between the points $m$ and $n$.

7. By the preceding we see that in order that there may be an equilibrium of a system of voussoirs, it is necessary that the components P and Q should satisfy four inequalities, which are to be verified for all the joints of the arch. Hence there exists certain limits between which the values of P and Q must be found. If these conditions do not contradict each other, and if the values of the components P and Q be such as to satisfy them, the equilibrium can subsist in the proposed system of voussoirs; and if we conceive the last joint MN applied against a fixed plane as the first joint AB is, and the system submitted to the action of forces which are applied to the voussoirs, we may rest assured that motion will not ensue.

*Plate Bande.*

8. The plate bande has its extrados and intrados, two parallel straight lines, and differs from all other arches, inasmuch as the joints are not perpendicular to the intrados.

Let ABMN represent half the plate bande, AM$=a$, the thickness MN$=t$; the inclination of the extreme joint AB with the vertical $=\theta'$, the weight of a unit of mass$=\pi$, and retaining the same notation as in art. 1, the area of the trapezoid M$mn$N is equal the sum of the areas of the rectangle MN$om$ and the triangle $mno$,




$$\text{area of rectangle} = (a-x)\,t$$

$$\text{area of triangle} = \tfrac{1}{2}\,t^2 \tan.\,\theta$$

$\therefore$ the weight of the trapezoid $=\pi\left\{(a-x)\,t+\tfrac{1}{2}\,t^2 \tan.\,\theta\right\}$

Abstracting from the effects of friction and cohesion, and making, in art. 5, P=0, H=0,

$$W=\pi\left\{(a-x)\,t+\tfrac{1}{2}\,t^2 \tan.\,\theta\right\};\ \text{we have}$$

$$\tan.\,\theta=\frac{2\pi\,(a-x)\,t}{2\,Q-\pi\,t^2}$$

which gives for the last joint AB

$$\tan.\,\theta'=\frac{2\pi\,a\,t}{2\,Q-\pi\,t^2}$$

hence $$Q = \frac{\pi(2at + t^2 \tan.\theta')}{2 \tan.\theta'}$$

$$\therefore \frac{\tan.\theta}{\tan.\theta'} = \frac{a-x}{a}.$$

This last equation shows that if the joints AB and $mn$ were produced, they would intersect the vertical in the same point o; hence the condition of equilibrium established in art. 5 will be satisfied if all the joints produced shall pass through the same point.

9. The conditions of art. 6 become for the plate bande where $y=0$, $y'=t$, $b=0$, $b'=t$, $P=0$, $\omega=0$,

$$Qt > W(\alpha - x),$$
$$-Qt < W(\alpha - x'),$$

for any joint $mn$; but for the extreme joint AB we have

$$Qt > \text{moment of trapezoid round A},$$
$$-Qt < \text{moment of trapezoid round B}.$$

The moment of the trapezoid round A is the difference of the moments of the triangle ABC, and the rectangle ACNM,

$$\text{moment of rectangle} = \tfrac{1}{2} a \,.\, \pi a t$$
$$\ldots\ldots \text{triangle} = \tfrac{1}{3} t \,.\, \tan.\theta' \,.\, \tfrac{1}{2} \pi t^2 \tan.\theta'$$
$$\therefore \text{moment of trapezoid} = \tfrac{1}{2} \pi a^2 t - \tfrac{1}{6} \pi t^3 . \tan.^2 \theta'.$$

But the moment of the trapezoid round B is equal to the sum of the moments of the triangle and rectangle.

$$\text{Moment of rectangle} = (\tfrac{1}{2} a + t \,.\, \tan.\theta')\, at\pi$$
$$\ldots\ldots \text{triangle} = \tfrac{2}{3} t \,.\, \tan.\theta' \,.\, \tfrac{1}{2} \pi t^2 . \tan.\theta'$$
$$\therefore \text{moment of trapezoid round B is}$$

$$(\tfrac{1}{2}a+t\,.\,\text{tan.}\,\theta')\,at\pi+\tfrac{1}{3}\pi t^3.\,\text{tan.}^2\theta'.$$

Hence the above conditions for the joint AB

$$Q\,t > \tfrac{1}{2}\pi a^2\,t-\tfrac{1}{6}\pi t^3.\,\text{tan.}^2\theta',$$
$$-Q\,t < \tfrac{1}{2}\pi at\,(a+2t\,.\,\text{tan.}\,\theta')+\tfrac{1}{3}\pi t^3.\,\text{tan.}^2\theta'.$$

The second inequality shows that the plate bande ABNM cannot turn round the upper arris B, since whatever positive value be given to Q, this condition will be satisfied.

The first inequality shows that ABNM cannot turn on the lower arris A, and it becomes, by substituting for Q its value from art. 8,

$$at > \tfrac{1}{2}\,(a^2-t^2)\,\text{tan.}\,\theta'-\tfrac{1}{6}\,t^2\,\text{tan.}^3\theta'$$

The extreme limit for $\theta'$ may be found by solving the equation,

$$\text{tan.}^3\theta'-3\left(\frac{a^2-t^2}{t^2}\right)\text{tan.}\,\theta'+\frac{6a}{t}=0.$$

If we take $\theta'=45°$, then since tan. $45°=1$, the equation becomes, if we make $t=1$,

$$1-3\,(a^2-1)+6a=0$$

this solved gives $a=1+\sqrt{\frac{7}{3}}=2{\cdot}527$, or the whole span $5{\cdot}054$, which shows that the utmost limit to which the span can be extended is about five times its thickness; but that the equilibrium may be stable we must have $a<2{\cdot}527$, or the whole span $<5{\cdot}054$.

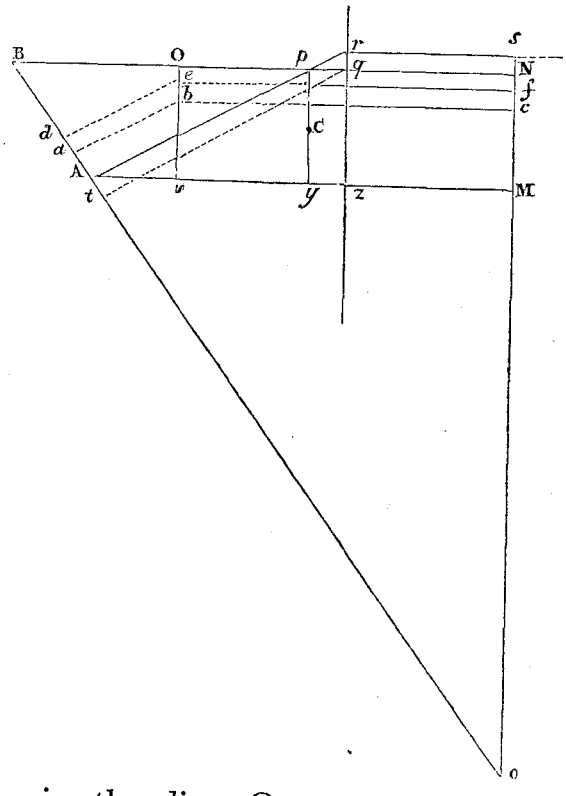
If $\theta'=30°$, then $\text{tan.}\,\theta'=\dfrac{1}{\sqrt{3}}$ and the equation becomes

$$\frac{1}{3\sqrt{3}}-3\,(a^2-1)\,.\,\frac{1}{\sqrt{3}}+6a=0$$

which solved gives $a=3{\cdot}76$, or the whole span $7{\cdot}52$;

here the utmost limit gives about seven and a-half times the thickness for the extent of the span.

10. If a vertical line $py$ be drawn through the centre of gravity, and from $p$, $p$A be drawn, then if AB be drawn perpendicular to $p$A, it will represent the extreme limit for the inclination of the joint; for we can draw from no other point in the line $py$, except $p$, two lines, the one perpendicular to MN, and the other to AB. Draw any two lines, O$v$ and $rqz$, one on each side of the vertical through the centre of gravity; from A produce A$p$ to $r$, then the other perpendicular $rs$ will evidently fall above N; and if from the point $q$ the line $qt$ be drawn parallel to A$pr$, it will fall below A, or in BA produced, and from any other point in the line $qz$ below $q$, the perpendicular will fall farther below A; also from any point above $r$ in this line, the perpendicular will fall farther above N:




but, on the contrary, in the line O$v$, we can draw from any point between O and A$p$, perpendiculars to AB and MN, such for instance as $ab$, $bc$, $de$, $ef$, &c. From this it follows that the equilibrium will be stable if the

centre of gravity should fall in any vertical line $Ov$, within the triangle $ABp$; strict but easily deranged if it fall in $py$; and should it fall in any vertical line beyond this as $qz$, the arch cannot stand independent of friction and cohesion.

By art. 4 the pressure exercised perpendicularly on any joint $mn$ is

$$T=(P+W)\sin.\theta+(Q+\omega)\cos.\theta,$$

which becomes for the plate bande where $P=0$, $\omega=0$

$$T=W\sin.\theta+Q\cos.\theta,$$

and for the last joint AB, by substituting for W and Q their values, viz., $W=\pi(at+\frac{1}{2}t^2.\tan.\theta')$ and

$$Q=\frac{\pi(2at+t^2.\tan.\theta')}{2\tan.\theta'}$$

we have

$$T=\pi\left(\frac{at}{\sin.\theta'}+\frac{t^2}{2\cos.\theta'}\right).$$

11. If the thickness of the plate bande be very small, compared with the span; then for the utmost limit, the equation $at-\frac{1}{2}(a^2-t^2)\tan.\theta'+\frac{1}{6}t^2.\tan.^3\theta'=0$ becomes, neglecting the quantities that involve $t^2$, which will be very small,

$$at-\tfrac{1}{2}a^2\tan.\theta'=0$$

$$\therefore\ \tan.\theta'=\frac{2t}{a}.$$

Therefore

$$Q=\frac{\pi at}{\tan.\theta'}=\tfrac{1}{2}\pi a^2;$$

also the expression for the perpendicular pressure on AB becomes

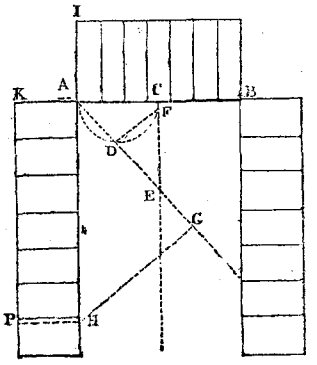
$$T=\frac{\pi at}{\sin.\theta'}$$

The following problem is taken from Professor Moseley's "Illustrations of Science," page 211.

"TO FIND THE GREATEST HEIGHT OF THE PIERS, OF A GIVEN WIDTH, WHICH WILL SUPPORT A STRAIGHT ARCH OF GIVEN DIMENSIONS.

12. Let AIB be the straight arch to be supported, and AK the given width of the piers.

Divide AB into two equal parts in C: upon AC describe a semicircle, and measure off AD equal to AK, so as to cut the circumference of this semicircle in D: produce AD, and let it intersect the vertical line through C in E: measure off EF equal to AI, and AG equal to AB: join DF, and draw GH parallel to DF; then AH will be the extreme height of the pier. Being of any less height, it will stand firmly; being of any greater, it will be overthrown."



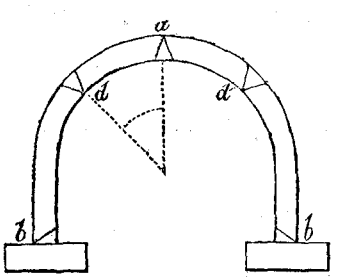

13. To establish the theory of arches, La Hire and other geometers supposed that they always break at points equally distant from the key-stone and the springings, and that the higher portion acts like a wedge against the joints of rupture, and tends to separate the lower portions; consequently, to obtain the thickness which the piers or abutments must have in order to resist this thrust, they endeavoured to find the pressure which is exerted perpendicular at one of the joints of rupture by making the moment with respect to the exterior edge

of the base of the pier equal to the moment of the half arch and its pier with respect to the same edge, and thus obtained an equation of equilibrium which gave the thickness of the pier.

The experiments of Danisy, and afterwards those of Boistard, have shown that arches may give way by sliding as La Hire supposes, but that the rupture more frequently occurs by means of rotation on the edges of the joint of rupture.

They have also proved that the position of the intermediate joints of rupture varies according as the force exerted by one part of the arch is greater than that exerted by the other.

14. If the force exerted by the upper portion be greater, that part tends to descend by means of the lower parts giving way, and the arch will break, as shown in the figure, that is to say, in five places, and the four portions of the arch turn round the edges $b$, $d$, $a$, $d'$, $b'$; but if on the contrary the force exerted by the lower portions is the greater, the arch breaks as shown in the figure, page 58.




Arches, therefore, give way at the key-stone, at the springings, and at the intermediate points; but as it has been before observed, the position of the last points of rupture varies according as the force exerted by the upper or lower extremities is greater.

Coulomb, profiting by the experiments of Danisy, was the first that considered the theory of arches with respect both to the sliding of the voussoirs on each other, and

also rotation round the upper or lower edges of the joints. He showed that the theory of La Hire was insufficient; and first observed that an arch may break into four parts instead of three. (See Mémoires Présentées, &c. tome vii. pp. 381 and 382).

15. Since the time of Coulomb succeeding writers have contented themselves with developing his theory: among the most successful may be mentioned Colonel Audoy in No. 4, and Mr. Petit in No. 12, Mémorial de l'Officier du Génie. Also perhaps one of the clearest expositions of Coulomb's theory may be found in Navier's excellent work, entitled *Résumé des Leçons données à l'Ecole des Ponts et Chaussées.*[2] Garidel's *Tables des Poussées des Voûtes en Plein Cintre*, and *Mémoire sur la Stabilité des Voûtes*, by M. G. Lamé and E. Clapeyron, may be consulted with much advantage. These two latter eminent engineers gave a number of transcendental equations for determining the points of rupture, but they have given the investigation as regards rotation only, and experience proves that this kind of rupture is most to be dreaded.

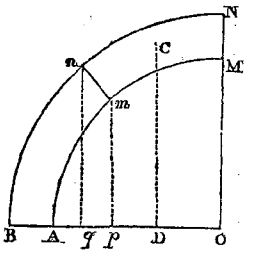
16. To proceed with the exposition of Coulomb's theory we may consider as given—the span, the rise, the curve of intrados, the height of the piers or abutments, the distribution of the weights which the arch ought to sustain, and the thickness of the arch at the key; this thickness is generally determined from the example of the most perfect constructions, similar to the one about

[2] To this admirable work we are greatly indebted, and consider that it should be used in every school that is at all interested in the progress of science applied to the arts.

to be projected. The equilibrium of the arch is maintained either by loading the parts more that tend to be raised, or by giving more thickness to these parts.

Let us suppose the arch to be divided into two equal parts at the key, the action of the weights supported by the arch exercises a pressure perpendicular to this joint between the two halves; we may therefore suppress one half of the arch, and replace it by a horizontal force equal to the pressure it exercises.

17. Let ABNM represent a semi-arch, and $mn$ any joint where the rupture may be supposed to take place. Let OM and ON$=b$, $b'$; $x$, $y$ the co-ordinates of the point $m$; $x'$, $y'$ those of the point $n$; $z$ the length of the joint $mn$; $\theta$ the angle which this joint forms with the vertical; $\alpha=$AD the distance of the point A from the vertical passing through the centre of gravity of this portion.




T the normal pressure on the joint $mn$, and for the rest the same notation as in art. 1.

18. Supposing that the rupture of the arch takes place by the sliding of the voussoirs along the planes of the joints, we have by art. 2

$$F=\frac{W(\cos.\theta-f\sin.\theta)-rz}{\sin.\theta+f\cos.\theta}.* \quad . \quad . \quad (1)$$

The values of F in this equation must be calculated for all the joints in the semi-arch, since by varying the

* The friction which is proportional to the perpendicular pressure on the joint will be

$$f(F\cos.\theta+W\sin.\theta)$$

and the cohesion being proportional to the length of the joint, is $rz$.

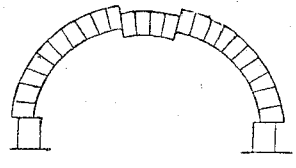
angle $\theta$, we will have different values for F. The greatest of these values ought to be taken for the horizontal thrust, or the pressure which the two halves of the arch exercise against each other at the key.

The force F, which would be sufficient to cause the portion MN$nm$ of the arch to slide upwards in the direction $mn$, may be thus expressed:

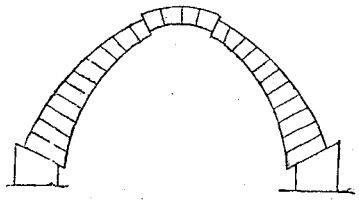
$$F = \frac{W(\cos.\theta + f\sin.\theta) + r\varepsilon}{\sin.\theta - f\cos.\theta} \quad . \quad . \quad . \quad . \quad (2)$$

The values of F being calculated in the same way for all the joints in the semi-arch, the least value must be greater than the horizontal thrust, or the equilibrium requires that the maximum of equation (1) must be less than the minimum of equation (2).

19. According to the forms and proportions generally given to arches, the joints of rupture which will give the maximum of equation (1) will be found in the haunches of the arch. The joint which gives the minimum value of equation (2) is near the springings; here the arch has a tendency to give way, as in the annexed figure, the upper parts sliding downwards, and thereby forcing the lower parts upwards along the springing line.



It may also happen that the joint which gives the maximum of equation (1) may be at or near the springing line, whilst that which corresponds to the minimum of equation (2) is near the crown; then the arch will be apt to give way as in the annexed figure.



20. Now that the rupture cannot take place by the voussoirs turning on the upper or lower edges of the joints, by art. 1, the horizontal force F, applied at the point N, necessary to prevent the portion $mn$NM of the arch from turning from the top to the bottom on the lower edge or arris $m$, may be thus expressed:

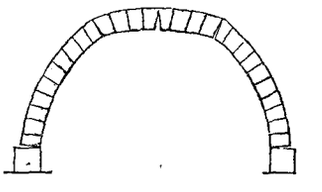
$$F = \frac{W(\alpha - x) - \frac{1}{3} R z^2}{b' - y} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The values of (2) being calculated for all the joints in the semi-arch, the greatest of these values ought to be taken for the horizontal thrust. Also, the horizontal force F applied at N, which would be sufficient to cause the portion $mn$NM to turn upon the upper arris $n$, may be expressed by the equation

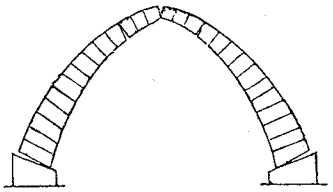
$$F = \frac{W(\alpha - x') + \frac{1}{3} R z^2}{b' - y'} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The values of (4) being also calculated for all the joints in the semi-arch, the least value must be taken greater than the horizontal thrust; that is, the equilibrium requires that the minimum of equation (4) must be greater than the maximum of equation (3).

From the above the maximum of equation (3) is given by a joint of rupture near the key, and the minimum of equation (4) given by a joint of rupture near the springings. Then the arch is apt to break, as represented in the annexed figure, the upper parts turning inwards round the arris $m$, forcing the lower parts of the arch to turn round the exterior arris of the joints near the springings.



21. But, at page 80, "Papers on Bridges," we have seen



that the arch may give way, as in the figure, the lower parts turning inwards, and thereby forcing the upper parts outwards. For the equilibrium for this kind of rupture, the force F applied at M, which would prevent the portion of the arch $mn$NM from turning on the edge or arris $m$, may be thus expressed:

$$F = \frac{W(\alpha - x) - \frac{1}{3} R z^2}{b - y} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The maximum of this equation must be taken for the horizontal thrust of the arch.

Also the force F applied at M, figure, page 36, which would cause the same portion to turn on the arris $n$, is

$$F = \frac{W(\alpha - x') + \frac{1}{3} R z^2}{b - y'} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

For the equilibrium in this case, the maximum of equation (5) must be less than the minimum of equation (6).

22. In art. 18 we have supposed the rupture to take place only by sliding, and in arts. 20 and 21, that it should be effected by rotation. The most general case is where the arch is apt to give way as is represented in the figure, page 38; it also sometimes happens that whilst the upper parts, as here represented, are descending, they force the lower parts outwards, as in the figure, page 37. It is clear that to prevent this motion the maximum of the expression equation (3) must be less than the minimum of (2).

For every possible combination that can take place, we must, in the first instance, have the equations (1) and (3), throughout the whole arch, less than the equations

(2) and (4); and in the next place, the equations (1) and (5) less than (2) and (6).

23. When the arch is built or supported on piers or abutments the preceding formulæ may be here applied, if we consider the piers or abutments to form part of the arch; but this supposes that the stones or voussoirs of the piers are sufficiently long to extend throughout the whole thickness, for the theory here treated of is founded on the hypothesis that the arch can only give way at the joints, either by sliding on their planes or turning on the edges.

24. From the preceding we may proceed to give the conditions of equilibrium in every possible case. 1st. To find the horizontal thrust F applied at N by supposing different positions of the joint of rupture *mn*, in the haunches of the arch, and stopping at the position for which the expression art. 21

$$F = \frac{W(\alpha - x) - \frac{1}{3}Rz^2}{b' - y},$$

or by neglecting the effects of cohesion,

$$F = \frac{W(\alpha - x)}{b' - y}$$

gives the greatest value.

We must next ascertain whether the horizontal thrust thus determined be sufficient to cause either the whole or a part of the semi-arch ABNM to turn on the exterior arris. Now it is clear that the horizontal thrust can more easily force the whole semi-arch to yield, either by sliding or rotation, than it can a part of that semi-arch, where the inclination of the planes of the joints to the horizon is greater.

For the equilibrium to exist, supposing the rupture in the first joint AB, we must see that the expression

$$F = \frac{W(\alpha - x') + \frac{1}{3}Rz^2}{b' - y'},$$

or neglecting the effects of cohesion,

$$F = \frac{W(\alpha - x')}{b' - y'},$$

calculated for this joint, is greater than the horizontal thrust.

We must also see whether the horizontal thrust can make either a part or the whole of the semi-arch slide on the planes of the joints; that is, we must show that the expression

$$F = \frac{W(\cos.\theta + f \sin.\theta) + rz}{\sin.\theta - f\cos.\theta},$$

or neglecting cohesion,

$$F = \frac{W(\cos.\theta + f\sin.\theta)}{\sin.\theta - f\cos.\theta},$$

calculated for any joint whatever near the springings, is greater than the horizontal thrust. If this joint were horizontal, the preceding expression would become

$$F = Wf + rz,$$

or neglecting cohesion,

$$F = Wf.$$

25. In the note, page 88, "Papers on Bridges," we observed that the theory there given gave the same results as far as rotation was concerned as the theory of Coulomb, but that the latter was much simpler in form. We shall now reproduce the formulæ given in the Papers from that of Coulomb.

Equation (3) page 38, $F=\frac{W(\alpha-x)}{b'-y}$ neglecting cohesion. Here $\alpha-x=DP$, the distance of a vertical through the centre of gravity of the portion $mn$NM from D, and $b'-y=EQ$, using the rotation given in the Papers, we have

$$Q=\mu\frac{DP}{EQ}.$$

Now by equation (4) the force Q, applied at E to turn the whole semi-arch round the arris K, may be thus determined.

The moment of the upper portion of the arch round K is $\mu$ (DP+KR).

The moment of the lower portion is $\nu$. KS, the sum of these moments is the whole moment of the weight of the semi-arch round the point or arris K.

Also the moment of Q round K=Q.KX=Q (EQ +KU). Now in order to have a strict equilibrium these moments must be equal

$$Q\,(EQ+KU)=\mu\,(DP+KR)+\nu.\,KS$$

$$\therefore\ Q=\frac{\mu\,(DP+KR)+\nu.\,KS}{EQ+KU}.$$

Hence to ensure the stability we must have

$$\mu\frac{DP}{EQ}<\frac{\mu\,(DP+KR)+\nu.KS}{EQ+KU},$$

or by reduction

$$\mu\frac{DP\cdot KU}{EQ}<\mu.KR+\nu.KS,$$

but the triangles DMP and DEQ are similar

$$DP : PM\ (=FQ) : : DQ : EQ$$

$$\therefore \mathrm{DP} = \frac{\mathrm{DQ} \cdot \mathrm{FQ}}{\mathrm{EQ}};$$

this substituted in the above gives

$$\mu . \frac{\mathrm{FQ}}{\mathrm{EQ}} . \frac{\mathrm{DQ}}{\mathrm{EQ}} . \mathrm{KU} < \mu . \mathrm{KR} + \nu . \mathrm{KS},$$

which is the same expression as given at page 84, "Papers on Bridges."

From this we see clearly the superiority of the theory of Coulomb over the complicated theory of rotatory levers, having deduced the same expression from the most simple and evident principles; besides, as we have before observed, the former is more general than the latter, as it takes into account the tendency of the voussoirs to slide on the planes of their joints.

26. M.M. Lamé and Clapeyron find the joint of rupture by making $\frac{m\,a}{h}$ a maximum, where $m$ is the mass above that joint, $a$ the distance between the point of rupture, and the vertical passing through the centre of gravity and $h = \mathrm{EQ}$, (see figure, page 81, "Papers on Bridges:") they give

$$\mathrm{H}\left(\frac{\mathrm{MA}}{\mathrm{H}} - \frac{m\,a}{h}\right),$$

a maximum where H, M, and A are the corresponding quantities to the above, taking both the arch and its piers into consideration; and since H and $\frac{\mathrm{MA}}{\mathrm{H}}$ are constant quantities, the above will be a maximum when $\frac{m\,a}{h}$ is a maximum. For semicircular arches with parallel extrados let R and $r$ be the radii of bases of the cylin-

ders of extrados and intrados, $r\theta$ the arc between the middle of the key-stone and the joint of rupture, then $h=$ R$-r$ cos. $\theta$; $m=\frac{R^2-r^2}{2}.\theta$ and $a=r$ sin. $\theta-x$, $x$ being the distance of the centre of gravity of the mass $m$ from the vertical passing through the middle of the key. Let us conceive the mass $m$ to be divided into infinitely small elements by planes passing through the common axis of the cylinders, making between them the constant angle $d\theta$, the centre of gravity of each of the equal elements will be distant from the axis of the cylinders by a constant quantity $r'$, which by the property of the centre of gravity is thus determined,

$$\frac{R^2\,d\theta}{2}.\frac{2R}{3}-\frac{r^2\,d\theta}{2}.\frac{2r}{3}=\left(\frac{R^2\,d\theta}{2}-\frac{r^2\,d\theta}{2}\right)r',$$

$$\text{whence } r'=\tfrac{2}{3}\,\frac{(R^3-r^3)}{(R^2-r^2)};$$

$r'$ being known we must determine $x$,

$$x\left(\frac{R^2-r^2}{2}\right)\theta=\int r'\sin.\theta\left(\frac{R^2-r^2}{2}\right)d\theta,$$

$\therefore \theta x=r'$ (A$-$cos. $\theta$), the value of the constant A is unity, for the integral ought to vanish when $\theta=0$; we have then

$$a=r\sin.\theta-\frac{2\,(R^3-r^3)}{3\,(R^2-r^2)},$$

and by substituting the values of $m$, $a$, and $h$ in $\frac{ma}{h}$, we have

$$\frac{ma}{h}=\frac{\frac{(R^2-r^2)}{2}r\theta\sin.\theta-\frac{(R^3-r^3)}{3}(1-\cos.\theta)}{R-r\cos.\theta}, \quad . \quad . \quad (1)$$

which must be a maximum.

Putting $u$ for $\frac{m\,a}{h}$, and differentiating for $\theta$, we have

$$\frac{du}{d\theta}=\frac{\left\{\left(\frac{R^2-r^2}{2}\right)r(\sin.\theta+\theta\cos.\theta)-\left(\frac{R^3-r^3}{3}\right)\sin.\theta\right\}\left\{R-r\cos.\theta\right\}-\left\{\left(\frac{R^2-r^2}{2}\right)r\,\theta\sin.\theta-\left(\frac{R^3-r^3}{3}\right)(1-\cos.\theta)\right\}r\sin.\theta}{(R-r\cos.\theta)^2}=0$$

$$\therefore\left(\frac{R^2-r^2}{2}\right)r\left\{(\sin.\theta+\theta\cos.\theta)\,(R-r\cos.\theta)-r\,\theta\sin.^2\theta\right\}=\left(\frac{R^3-r^3}{3}\right)\left\{\sin.\theta\,(R-r\cos.\theta)-(1-\cos.\theta)\,r\sin.\theta\right\}$$

$$\left(\frac{R^2-r^2}{2}\right)r\left\{\sin.\theta\,(R-r\cos.\theta)+R\,\theta\cos.\theta-r\,\theta\cos.^2\theta-r\,\theta\sin.^2\theta\right\}=\left(\frac{R^3-r^3}{3}\right)\sin.\theta\,(R-r)$$

$$R-r\cos.\theta-\frac{\theta}{\sin.\theta}(r-R\cos.\theta)=\frac{2\,(R^3-r^3)}{3\,(R+r)r}$$

$$R-r\cos.\theta-z\,(r-R\cos.\theta)=\frac{2\,(R^3-r^3)}{3\,(R+r)\,r},\text{ putting }z\text{ for }\frac{\theta}{\sin.\theta},\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\quad(2)$$

which has for its root the arc $\theta$, corresponding to the joints or arris of rupture.

From the form of this equation we see that the position of the point of rupture depends on the relation of the radii R and $r$, or, which is the same, on the relation of the thickness of the arch to the diameter.

This equation, being transcendental, can only be solved by approximation. In the following table we have the different arcs $\theta$ from 45° to 60° with the corresponding values of sin. $\theta$, cos. $\theta$, and $z=\frac{\theta}{\text{sin. }\theta}$.

| Values of $\theta$. | Values of sin. $\theta$. | Values of cos. $\theta$ | Value of $z=\frac{\theta}{\text{sin. }\theta}$ |
|---|---|---|---|
| 45° or 0·7854 | 0·7071 | 0·7071 | 1·1133 |
| 46 0·80285 | 0·7193 | 0·6947 | 1·1161 |
| 47 0·8203 | 0·73134 | 0·6820 | 1·1216 |
| 48 0·83776 | 0·7431 | 0·6691 | 1·1273 |
| 49 0·8552 | 0·7547 | 0·65606 | 1·1332 |
| 50 0·87266 | 0·7660 | 0·6428 | 1·1392 |
| 51 0·8901 | 0·7771 | 0·6293 | 1·1453 |
| 52 0·9076 | 0·7880 | 0·6157 | 1·1517 |
| 53 0·9250 | 0·7986 | 0·6018 | 1·1582 |
| 54 0·9425 | 0·8090 | 0·5878 | 1·1649 |
| 55 0·9599 | 0·8191 | 0·5736 | 1·1719 |
| 56 0·9774 | 0·8290 | 0·5592 | 1·1790 |
| 57 0·9948 | 0·8387 | 0·5446 | 1·1862 |
| 58 1·0123 | 0·8480 | 0·5299 | 1·1937 |
| 59 1·0297 | 0·8572 | 0·5150 | 1·2014 |
| 60 1·0472 | 0·8660 | 0·5000 | 1·2090 |

The radius is always supposed equal to unity.

Let us take for an example the case of an arch whose thickness is $\frac{1}{16}$th of its diameter, then $\frac{R}{r}=\frac{9}{8}$, and the equation (2) becomes

$$(9-8\text{ cos. }\theta)-z\,(8-9\text{ cos. }\theta)=1\cdot0637 \quad . \quad . \quad . \quad (3)$$

Suppose $\theta=45°$, and we get 1·5218, which is too large; also if $\theta=0$, we have 2, which is also too great: hence we conclude that the angle of rupture is between 45° and 90°. If $\theta=60$, we get a quantity ·7686, too small in the left hand member of equation (3); therefore the real value is between 44° and 60°.

As the two preceding suppositions have given results which differ from the second part of the equation (3) of ·5 and ·3 respectively, the true value appears to be nearer 60° than 45°, and from the above relation it appears that $\theta$ is about 55°; this substituted gives 1·0859, which is still a little too great, which shows that $\theta$ is greater than 55°; and since $\theta=56°$ gives a result 1·0282, a value which is too small, we may therefore easily find by interpolation $\theta=55°\ 23'$; for taking the difference for 1° or 60′, and the difference between result for 50°, and the second member of equation (3), we have

$$0577 : 60 : : \cdot 0222$$

$$\begin{array}{r} 60 \\ \hline \cdot 0577)\ 1\cdot 3320\ (23' \\ 1\ 154 \\ \hline 1780 \\ 1731 \\ \hline 49 \end{array}$$

If $\theta=55°\ 23'$ be substituted, it will give in the first member of equation (3) 1·064, which differs from the second member only by the small quantity ·0003.

Thus the circular arch, whose thickness is constant and rise $\frac{1}{16}$th of the span, has the point of rupture situated 55° 93′ from the middle of the key-stone, or 34° 37′ from the springings.

27. The theory of Coulomb, as originally given by that

illustrious mathematician, and subsequently reproduced by Navier, which is nearly the same as we have here given, supposes a separate discussion of the conditions of equilibrium of each particular voussoir, and establishes the required maxima and minima by a comparison of the various different results thus obtained. In this form it supposes an immense labour of calculation; and, after all, it determines only the conditions of equilibrium of an existing structure, lending its aid but indirectly, and with difficulty, to the engineer who would determine the form and dimensions of a proposed structure so as best to secure its stability. The subsequent labours of M.M. Audoy, Lamé and Clapeyron, Persy, Petit, Poncelet and Garidel, have however given to this theory a new form and character, embracing in the discussion many conditions of the equilibrium of the arch which lay before beyond its limits, and greatly diminishing the labour of its calculations. From the account which our limits permitted to give of it, it will appear that this new developement of the theory of Coulomb consists in the determination in terms of the inclination of any joint of the arch to the vertical of a function expressing the value of that horizontal thrust, which, being applied to the summit of the key-stone, will just prevent the semi-arch from turning inwards upon that joint. By the theory of Coulomb the maximum of this function determines the position of the joint of rupture.

Professor Moseley has shown that his theory, founded upon the discussion of the lines of resistance and pres-

sure, (the former of which he has been the first to introduce in the theory of statics,) and developing itself by a wholly independent method of analysis, so far as it embraces the same elements of the discussion with the theory of Coulomb, necessarily leads to the same results; and it is a remarkable verification of the formulæ given by this able mathematician, and of those deduced from the theory of Coulomb by the eminent individuals whose names we have mentioned, that, proceeding with methods of analysis so remote and so difficult, they have arrived at formulæ, which, when they refer to the same circumstances of equilibrium, are identical.

The formulæ arrived at by the French mathematicians as stated by Garidel, and made by him the foundation of the tables which he has calculated with so much labour and ingenuity, are the following; the notation being made to correspond with that of Professor Moseley's Paper. In the case in which the load rests on the extrados, and the arch is a complete segment, so that $\Theta = 0$:

Garidel. Tables de la Poussée des Voutes, p. 30.

$$\frac{P}{r^2}=\frac{3\left\{(1+\alpha)^2-1\right\}\Psi\sin.\Psi-2(1-\cos.\Psi)\left\{1+\alpha)^3-1\right\}}{6(\alpha+1-\cos.\Psi)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Garidel, p. 31.

$$\frac{P}{r^2}=\alpha\left\{\left(1+\frac{\alpha}{2}\right)\Psi\cot.\Psi-\alpha\left(\frac{1}{2}+\frac{\alpha}{3}\right)\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Petit. Mémorial de l' Officier du Génie, No. 12.

$$\cos.\Psi+\frac{\Psi}{\sin.\Psi}-(1+\alpha)\Psi\cot.\Psi=1+\alpha-\frac{2}{3}\frac{(1+\alpha)^3-1}{\alpha+2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

In the case of an arch where loading is bounded by a straight line inclined to the horizon at a given angle :

Garidel, p. 44.

$$0=-A+B\cos.\Psi-C\cos.^2\Psi+D\cos.^3\Psi+\frac{\Psi}{\sin.\Psi}-(1+\lambda)\Psi\cot.\Psi+E\sin.\Psi(1-\tfrac{2}{3}\sin.^2\Psi)-H\sin.\Psi\cos.\Psi \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Where $A=\frac{2}{3}+(1+\beta)(1+\alpha)(1-\alpha^2)+\alpha^2(1+\frac{2}{3}\alpha)(1+\lambda)$

$B=1+2(1+\beta)(1+\alpha)(1-\alpha^2)(1+\lambda)$

$C=(1+\alpha)\{(1+\beta)(1-\alpha^2)+(1+\alpha)(1-2\alpha)(1+\lambda)\}$

$D=\frac{2}{3}(1+\alpha)^2(1-2\alpha)$

$F=(1+\alpha)^2(1-2\alpha)\tan.\iota$

$H=(1+\alpha)^2(1-2\alpha)(1+\lambda)\tan.\iota$

When the direction of the pressure P is through the bottom of the key-stone :

Garidel, p. 44.

$$0=-\left\{\tfrac{2}{3}+(1+\alpha)(1+\beta)(1-\alpha^2)+\alpha^2(1+\tfrac{2}{3}\alpha)\right\}+\left\{\tfrac{1}{3}+(1+\beta)(1+\alpha)(1-\alpha^2)-\alpha^2(1+\tfrac{2}{3}\alpha)\right\}\cos.\Psi+2\left\{\frac{2\alpha^3}{3}+\alpha^2-\tfrac{1}{3}\right\}\cos.^2\Psi+\frac{\Psi}{\sin.\Psi}+(1+\alpha)^2(1-2\alpha)\sin.\Psi(\tfrac{1}{3}-\tfrac{2}{3}\cos.\Psi)\tan.\iota \quad (5)$$

Now, if in equation (7) of Professor Moseley's Paper, page 51, X and Y be taken=0, by which substitutions the more general case of equilibrium supposed in that equation will be reduced to the case of a complete arch of equal voussoirs without loading, and if this equation be then solved in respect to $\frac{P}{r^2}$, then equation (1) of the above will be reproduced. Equation (13) of Professor Moseley's Paper will, in like manner, give us equation (2) of the above. If in the above equations (3) and (4), $\iota$ be assumed = 0, and $\lambda$ taken = $\alpha$, the Professor's equations (19) and (20) will be obtained. These are the fundamental equations from which the Tables of Garidel are calculated; and they result alike from the theory of Coulomb and that of Professor Moseley.[2]

The discussion of the latter theory, however, embraces various elements which are not, we believe, to be found in any other.

It determines the conditions of equilibrium not only of the continuous segmental arch, but of the Gothic arch, and that under every variety of loading; not only for instance when the pressure of the load is vertical, but when its direction is inclined at any angle to the ho-

[2] M. Garidel has given the following approximate expressions for the angle of rupture and the thrust in the case of an unloaded arch with equal voussoirs; they are derived from the equations (1) and (2).

$$\Psi = 57°293 + \frac{49{\cdot}594\,(\alpha + 1{\cdot}69043)\,(\alpha - 0{\cdot}15371)\,(0{\cdot}99987 - \alpha)}{(\alpha+2)\,(\alpha+0{\cdot}4597)}$$

$$\frac{P}{r^2} = \frac{0{\cdot}1532\,\alpha\,(\alpha+1{\cdot}7106)\,(1{\cdot}4565-\alpha)}{\alpha+0{\cdot}4597}$$

rizon: as an illustration of this fact, let it be observed that equations (25, 26, 27) determine the conditions of equilibrium of an arch which sustains, either by its extrados or intrados, the oblique pressure of a fluid. The equations of the Professor's Paper not only determine what are the conditions of the stability of an arch, under a given loading, but what loading will give certain conditions of stability. They enable us, for instance, so to load a segmental arch as to bring its points of rupture to any given distance from its springing; they also determine what load accumulated near the crown would cause the arch to fall by the descent of the crown, and what at the haunches would cause its fall by the elevation of the crown. In respect to the direction of the pressure upon the key-stone, these formulæ include, in common with the French formulæ, the case in which (the arch being constructed without cement) this direction may be supposed to be through the summit of the key-stone; they determine also the actual direction of this pressure when by the interposed cement a mathematical adjustment of the joints is brought about, and when this direction is therefore through some point which intervenes between the top and bottom of the key-stone,—a determination which, we believe, has not been attempted by any other author. It is, however, principally to be remarked in respect to the theories of Coulomb and Professor Moseley, that the latter is general, embracing every case of the equilibrium of a system of bodies in contact, and including the arch as a particular case. Of these various applications, (others of which are given by the learned Professor,) are

the buttress, the pier, the straight arch or plate bande, the embankment, &c.

The formulæ given by the French mathematicians for determining the width of the pier of a given height is deducible from Professor Moseley's equations, substituting in equation (4) the values of the horizontal thrust, the weight of the arch, and the quantity $k$, as determined by the subsequent equations, and then solving equation (4) for K.

28. We shall now proceed to give Petit's formulæ and tables for calculating the stability of arches.

At page 38 we have shown that if the length of the arm of the lever, with which the horizontal thrust F acts $= y$, and the weight of the portion $mn$NM$=$W, and $\phi$ the length of the arm at which it acts, then we have

$$\mathrm{F} \,.\, y = \mathrm{W} \,.\, \phi \text{ or } \mathrm{F} = \frac{\mathrm{W}\,\phi}{y}.$$

We are ignorant of the point $m$ round which the rotation tends to take place, or in other words, we do not know the angle $\theta$ which the joint of rupture $mn$ makes with the vertical; but since the force F must be such that it shall keep any portion whatever of the arch in equilibrium upon the portion beneath it, it must be equal the greatest value that $\frac{\mathrm{W} \times \phi}{y}$ admits by the variation of the point of rupture.

In order to obtain this value, we must assume a point of the arch, or the angle $\theta$, which determines the position of the point, and find the corresponding value of F. We must next assume another value of $\theta$, and

find another value of F, and proceed in this way until we have found the maximum value of F. (See page 38.)

We must take the moment of this last force with respect to the edge *d*, or the outer edge of the given piers, and then put this moment, equal to the moment (M) of the half arch and its pier, and this equation will give (*e*) the thickness of the pier. Thus, if F′ be the maximum value of F, and L the arm at which it acts, we have the equation

$$M=F'L, \text{ which gives } e.$$

These are the two formulæ which M.M. Audoy and Petit have developed. The following are M. Petit's formulæ, relating to arches that are semicircular and have a parallel extrados.

*Semicircular arches with parallel extrados.*

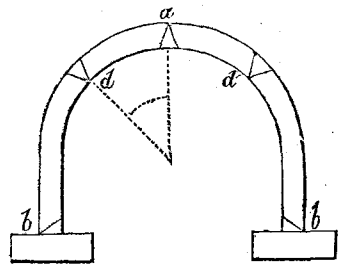
29. The formulæ for these arches are

$$\cos.\theta+(1-K\cos.\theta)\frac{\theta}{\sin.\theta}=K-\tfrac{8}{3}.\frac{K^3-1}{K+1} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$F'=r^2\left\{\tfrac{1}{2}(K^2-1)\left(1+\frac{\theta}{\sin.\theta}.\cos.\theta\right)-\tfrac{1}{3}(K^3-1)\right\} \quad . \quad . \quad . \quad (2)$$

$$\frac{e}{r}=-\tfrac{1}{4}\pi(K^2-1)\frac{r}{h}+$$

$$\sqrt{\tfrac{1}{16}\pi^2(K^2-1)^2.\frac{r^2}{h^2}+2\left\{Kc+\tfrac{1}{3}(K^3-1)-\tfrac{1}{4}\pi(K^2-1)\right\}\frac{r}{h}+2c} \quad (3)$$




In the preceding formulæ K is the ratio of the radius

of the extrados to that of the intrados or $\frac{R}{r}$; $\theta$ the angle of rupture corresponding to the maximum thrust; $F'$ the maximum value of the thrust F; $r$ the radius of the intrados; $h$ the height of the pier or abutment of the arch; $c$ the ratio of the maximum thrust, and the square of the radius or $c = \frac{F'}{r^2}$; $\pi$ the ratio of the circumference to the diameter.

As an example, we may determine the necessary thickness for the piers or abutments of a semicircular arch with parallel extrados, where the radius of the intrados $r = 16{\cdot}4$ feet, and the radius of extrados R $= 20{\cdot}99$ feet, and the height of the piers or abutments $6{\cdot}56$ feet.

With the Tables I. and II. of M. Petit this problem can be easily solved. We must first find the ratio $K = \frac{R}{r} = \frac{20{\cdot}99}{16{\cdot}4} = 1{\cdot}28$. In Table I. we find the corresponding value of $\theta = 62°30'$, and Table II. gives the following equation to solve

$$\frac{e}{r} = -{\cdot}5014 \cdot \frac{r}{h} + \sqrt{{\cdot}2520 \cdot \frac{r^2}{h^2} + {\cdot}0801 \cdot \frac{r}{h} + {\cdot}2738}$$

in order to determine the thickness of the abutments in the case of strict equilibrium.

The preceding equation is only equation (3) with the values $K = 1{\cdot}28$, $\pi = 3{\cdot}1416$, or $\frac{r}{h} = \frac{16{\cdot}4}{6{\cdot}56} = 2{\cdot}5$, and this equation gives $e = 2{\cdot}92$ for the case of strict equilibrium.

Without these tables we should first have supposed $\theta = 61°$ or $\theta = \frac{61 \times 3{\cdot}1416}{180} = 1{\cdot}06$, cos. $61° = {\cdot}4848$,

sin. $61° = \cdot 8746$; $K = 1 \cdot 28$; these numbers substituted in equation (1) we have $\cdot 9446 = \cdot 9592$; hence $61°$ does not satisfy the equation.

Next assume $\theta = 62°$, and, proceeding as before, we find $\cdot 95856 = \cdot 95920$; this value of $\theta$ therefore does not satisfy the equation.

Now assume $\theta = 63°$, and we find $\cdot 97093 = \cdot 95920$; here the first number is the greater, we therefore conclude that the real value of $\theta$ lies between $62°$ and $63°$.

Taking the difference $\cdot 97093 - \cdot 95856 = \cdot 01237$, which corresponds to $1°$ or $60'$, and also the difference $\cdot 95920 - \cdot 95856 = \cdot 00064$,

$$\cdot 01237 : 60' : : \cdot 00064 : x = 31';$$

hence $\theta = 62°\ 31'$; sin. $\theta = \cdot 8871$, cos. $\theta = \cdot 4615$, $\theta = \frac{62°\ 31'}{90} \cdot \frac{3 \cdot 1416}{2} = 1 \cdot 091$. These values substituted in equation (2) give $c = \frac{F}{r^2} = \cdot 135$, nearly the same as the value given by Table I., whence we may find the maximum thrust $F' = \cdot 135 \cdot (16 \cdot 4)^2 = 36 \cdot 31$ feet; lastly, this value of $c$, and those of $h$, $r$, $\pi$, and K, substituted in equation (3), give the thickness $e$, as has been found above. In the case where $h$ is infinite, the formula (3) would reduce to $\frac{e}{r} = \sqrt{2c}$, which gives for the limit of the thickness of the piers or abutments $e = 8 \cdot 528$ feet instead of $2 \cdot 92$.

By Colonel Audoy's method we may find the stability by multiplying the value of the thrust by $1 \cdot 9$, which gives the stability of arches calculated by the formula (3); by putting $1 \cdot 9\ c$ or $c + \cdot 9\ c$ instead of $c$, we have

$$\frac{e}{r} = -\tfrac{1}{4}\pi(K^2-1).\frac{r}{h} +$$
$$\sqrt{\left\{\tfrac{1}{16}\pi^2(K^2-1)^2.\frac{r^2}{h^2} + [1{\cdot}8.\,Kc + 2\,Kc + \tfrac{2}{3}(K^3-1) - \tfrac{1}{2}\pi(K^2-1)].\frac{r}{h} + 2c + 1{\cdot}8\,c\right\}}$$

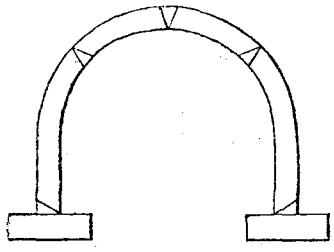
If we want to find the limit of thickness in the case of stability given by La Hire, the equation becomes $\frac{e}{r} = \sqrt{2c + 2\times{\cdot}9\,c} = \sqrt{3{\cdot}8\,c} = {\cdot}72$, whence $e = {\cdot}72 \times 1{\cdot}64 = 11{\cdot}8$ feet.

The voussoirs tend to slide on the surfaces of the inferior joints; hence there results a thrust sometimes more powerful than that of rotation. In semicircular arches with a parallel extrados the maximum thrust due to sliding is $G = {\cdot}15304\,r^2\,(K^2-1) = F'$, G being the horizontal force capable of preventing the sliding of any voussoir whatever which tends to descend upon the lower surface. It is by these means that Petit has found the numbers which are given in the 5th column of Table I. Thus, in the example $c = \frac{F'}{r^2} = {\cdot}15304\,(K^2-1) = {\cdot}0977$, a thrust which is less than the thrust from rotation.

If the former thrust were the greater, we should use it to determine the thickness of the piers. By examining Table I. we find the values of $c$ relative to sliding greater than those relative to rotation as far as $K = 1{\cdot}44$; thus, for those arches which give to K a value between 2·732 and 1·44, we must employ the values of $c$ relative to sliding for determining the thickness of the abutments.

The following figure represents the rupture of the arch

when the force exerted by the lower portions is greater than that by the upper.



*Semicircular arches with horizontal extrados.*

30. The formulæ relative to these arches are

$$F=\frac{r^2 \sin.^2\theta}{6(K-\cos.\theta)}\left\{K^2[6-3K-(3-2K)\cos.\theta]-\left(\frac{3\theta}{\sin.\theta}-\frac{1}{\cos.^2\frac{1}{2}\theta}\right)\right\} \quad (1)$$

and

$$\frac{e}{r}=-(K-\tfrac{1}{4}\pi)\frac{r}{h+Kr}+\sqrt{(K-\tfrac{1}{4}\pi)^2\left(\frac{r}{h+Kr}\right)^2+\left(2Kc-K+\frac{3\pi-4}{6}\right)\frac{r}{h+Kr}+2c\cdot\frac{h}{h+Kr}} \quad (2)$$

The first gives the thrust due to rotation, and the second the thickness $e$ of the piers.

And $G=r^2$ (·16391. $K^2$—·15206) (3), which gives the thrust due to the maximum of sliding. As an example, let us find the thickness of the abutments of an arch with an horizontal extrados, where the radius of intrados is $r=6{\cdot}56$ feet, and radius $R=7{\cdot}54$ feet, and the height of the piers $h=9{\cdot}84$ feet.

We shall have $K=\frac{R}{r}=\frac{7{\cdot}54}{6{\cdot}56}=1{\cdot}15$, and we might assume the angle of rupture $\theta=60°$ and find the corresponding value of F, and then make $\theta=61°$ and find the corresponding value of F by means of equation (1), and proceed in the same manner as at pages 55 and 56, until we found the maximum value of F; but, by Table III. we see that if $K=1{\cdot}15$, the corresponding value of the

angle of rupture is $\theta = 64°$, and $\frac{F'}{r^2} = c = \cdot 11895$ in the case of rotation.

In the case of sliding we obtain by the same Table $\frac{F'}{r^2} = \cdot 06471$.

According to what we have before observed, we shall take the first value of $c$ (it being greater than that due to sliding) to determine the thickness of the abutments; by substituting in (2) the values $K = 1 \cdot 15$, $\frac{F'}{r^2} = c = \cdot 11895$ and $\pi = 3 \cdot 1416$, we have

$$\frac{e}{r} = - \cdot 3646 \times \cdot 377 \sqrt{\cdot 01889 + \cdot 01044 + \cdot 13465},$$

hence $e = 1 \cdot 75$ feet for the case of strict equilibrium.

If $h$ were infinite, the equation (2) would reduce to $\frac{e}{r} = \sqrt{2c} = \sqrt{2 \times \cdot 11895}$; hence $e = 3 \cdot 2$ feet for the limit of the thickness.

If we wanted the practical stability, we ought to substitute $1 \cdot 9\ c$ for $c$ in equation (2). Similarly to obtain the limit of the thickness according to La Hire, we must substitute $1 \cdot 9\ c$ for $c$ in formula $\frac{e}{r} = \sqrt{2c}$, which gives $e = r \sqrt{3 \cdot 8\ c} = 4 \cdot 34$ feet nearly.

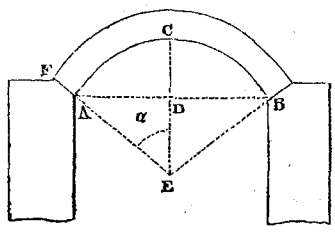
The Table III. shows that for values of K less than $1 \cdot 35$ we must consider in our calculation the thrust due to rotation, since it exceeds that due to sliding, and we must consider the latter thrust if K be equal or greater than $1 \cdot 35$.

If in the above example we want the thrust due to sliding, we obtain

$$\frac{F'}{r^2} = \cdot 16391\ K^2 - \cdot 15206 = \cdot 0647 \therefore F' = 2 \cdot 785.$$

*Arches in the form of a circular arc with parallel extrados.*

31. Besides the data necessary for the two preceding cases we must have the span AB, which we denote by L, and the rise CD, denoted by $f$.




Having L and $f$ given, we shall have, in order to determine the radius AE or $r$,

$$r = \frac{f}{2}\left(1 + \frac{L^2}{4f^2}\right)$$

and the angle AEC, which we denote by $a$, is given by the equation

$$\text{sin. } a = \frac{\frac{L}{f}}{\frac{1}{4} \cdot \frac{L^2}{f^2} + 1}.$$

When the span L and the angle $a$ are given, we shall have the radius $r$ by the formula $r = \frac{L}{2 \text{ sin. } a}$, and $f$ from $f = r\ (1 - \text{cos. } a)$, $r$ being known, and the thickness AF, we shall have EF = R, and therefore $\frac{R}{r} = K$. The Table I. relative to semicircular arches with a parallel extrados will give the angle of rupture $\theta$. It may happen,—1st. That this angle of rupture of the proposed arch, considered as a complete semicircle, is smaller than $a$ or half the angle at the centre; in this case the joint of rupture takes place between A and C, and the arch

ought to be considered relative to the horizontal thrust as a semicircular arch, and the Table I. will give the maximum thrust $F'=cr^2$ or $c=\frac{F'}{r^2}$. As for the thickness of the abutments it will be found by the equation

$$\frac{e}{r}=-\tfrac{1}{2}a\,.\,\frac{r}{h}(K^2-1)+$$
$$\sqrt{\left\{\tfrac{1}{4}a^2(K^2-1)^2.\frac{r^2}{h^2}+2[c(K-\cos.a)+\tfrac{1}{3}(K^3-1)(1-\cos.a)-\tfrac{1}{2}(K^2-1)\right.}$$
$$\left.a\sin.a]\,\frac{r}{h}+2c\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The limit of the thrust in the case of strict equilibrium is always given by $e=r\sqrt{2c}=\sqrt{2F'}$; that is, it is always equal to the square root of twice the horizontal thrust; and the stability, according to La Hire, is given by the equation

$$e=r\sqrt{2\times1\cdot9}\;c=r\sqrt{3\cdot8\,c}.$$

As an example, let us find the thickness of the abutments of an arch in the form of a circular arc with parallel extrados, where $a=62°$, the thickness AF=2·23 feet, L=19·68 feet, and $h=13\cdot12$ feet, we have

$$r=\frac{L}{2\sin.a}=\frac{19\cdot68}{2\times\cdot8829}=11\cdot145:f=r(1-\cos.a)=11\cdot145(1-\cdot4695)$$
$$=5\cdot9124;\ R=11\cdot145+2\cdot23=13\cdot375,\frac{R}{r}=\frac{13\cdot375}{11\cdot145}=1\cdot2=K,$$

and to this value of K corresponds the angle $\theta=59°\ 41'$ (Table I.), and $c=\cdot1114$, a value which is greater than the value of $c$ due to sliding; we shall therefore employ $c=\cdot1114$, in order to find the thickness of the abutments. We shall also have $a=\frac{62}{70}\cdot\frac{3\cdot1416}{2}=1\cdot0821$; all these values substituted in equation (4) give

$$\frac{e}{r}=-\cdot20223$$
$$+\sqrt{\cdot0409+2\left\{\cdot7305.c+\cdot1287-\cdot2101\right\}\cdot8425+2c},$$

whence $e = 11{\cdot}145 \times {\cdot}31 = 3{\cdot}45$ when the equilibrium is strict.

The limit of the thickness $e = r\sqrt{2c} = 11{\cdot}145 \times \sqrt{{\cdot}2228} = 5{\cdot}25$ feet.

For the stability according to La Hire we must put $1{\cdot}9\ c$ for $c$ in the above formula, and we shall have

$$\frac{e}{r} = -{\cdot}20223 + \sqrt{0.409 + 2\left\{{\cdot}7305 \times 1{\cdot}9\,c + {\cdot}1287 - {\cdot}2101\right\} \times {\cdot}8495 + 2 \times 1{\cdot}9\,c},$$

whence $e = r \times {\cdot}56 = 6{\cdot}23$ feet, and the limit $e = r\sqrt{2 \times 19\,c} = 7{\cdot}25$ feet.

It may also happen that the angle of rupture of the proposed arch, considered as a semicircle, is greater than half the angle at the centre or $a$, which usually occurs in practice; then the rupture takes place at the springings, and the thrust is given in this case by the equation

$$c = \frac{F'}{r^2} = \frac{\frac{1}{2}(K^2-1)\ a \sin.\ a\, r^3 - \frac{1}{3}(K^3-1)\ (1-\cos.\ a)\ ^3r}{K - \cos.\ a}.$$

Thus L and $f$ being given we have the ratio $\frac{L}{f}$, and the formula $\sin.\ a = \dfrac{\frac{L}{f}}{\frac{1}{4} \cdot \frac{L^2}{f^2} + 1}$ gives $a$; having a sin. $a$, cos. $a$, K and $r$, we obtain $c$, and the formula (4) will give the thickness of the abutments.

The Table IV. of M. Petit gives the thrust for those arches whose rise $f$ is the 4th, 5th, 6th, 7th, 10th, and 16th part of the span L, and for the different values of K; these are the kinds of arches most in use. We find also in these Tables, and for each of these cases, the

value of $a$, and that of $r$, which corresponds with it. Let $\theta$ be greater or less than $a$, formula (4) must be used to find the thickness of the abutments; only if $\theta$ be greater than $a$, we must take the values of $c$ found in Table IV.; and if $\theta$ be less than $a$, those in Table I. As an example, let us find the thickness of the abutments of an arch in the form of a circular arc with parallel extrados, where the angle at the centre $= 70°$ or $a = 35°$, and where the span $L = 49{\cdot}2$, and height $h = 13{\cdot}12$, we shall have $a = \frac{35}{90} \cdot \frac{3{\cdot}1416}{2} = {\cdot}6109$; sin. $a = {\cdot}5735$ nearly, cos. $a = {\cdot}8191$, $f = r\ (1 - \text{cos. } a) = {\cdot}1809\ r$; $r = \frac{L}{2 \text{ sin. } a} = \frac{L}{1{\cdot}147}$ $\therefore$ $L = 1{\cdot}147\ r$; $\frac{L}{f} = \frac{1{\cdot}147\ r}{{\cdot}1809\ r} = 6{\cdot}34$.

Let us also suppose the thickness of the arch $= 3{\cdot}28$ feet, $R = r + 3{\cdot}28$; $r = \frac{L}{1{\cdot}147} = \frac{49{\cdot}2}{1{\cdot}147} = 42{\cdot}89$, and Table I. gives for $K = \frac{46{\cdot}17}{42{\cdot}89} = 1{\cdot}076$, an angle of rupture $= 49°48'$, which is greater than the angle $a$; hence the thrust must be found by Table IV. Here $K = 1{\cdot}076$ is between $K = 1{\cdot}07$ and $K = 1{\cdot}08$; and $\frac{L}{f} = 6{\cdot}34$ is between $\frac{L}{f} = 6$ and $\frac{L}{f} = 7$, we may find by proportion the value of $c = \frac{F'}{r} = {\cdot}04681$.

With this value and the others given above we must proceed as in the former example, and by means of equation (4) determine the thickness of the abutments for a strict equilibrium. And for the stability according

to La Hire we shall find the limit of the thickness by the formula $e = r\sqrt{3{\cdot}8\ c}$.

It still remains for us to calculate the thrust due to sliding, in order to substitute it in formula (4), in the case where it is more powerful than that due to rotation.

If the half angle at the centre, or $a$, be greater than 26°, the horizontal thrust due to sliding is calculated by the formula

$$F' = {\cdot}15304\ (K^2-1)\ r^2.$$

If the half angle $a$ is less than 26°, we must substitute its value instead of $\theta$ in the formula

$$G = \tfrac{1}{2}\ r^2\ (K^2-1).\frac{\theta}{\text{tan. }(\theta+30°)},$$

and we shall have the thrust due to sliding on the joint at the springings. A horizontal interline drawn in the columns indicates for all the Tables the value of K where the one thrust exceeds the other.

*Arches in the form of a circular arc with horizontal extrados.*

32. After finding the radii R and $r$ as before, and consequently K, by the formula

$$F = \frac{r^2 \text{ sin.}^2\theta}{6\,(K-\cos.\theta)}\left\{K^2\,[6-3K-(3-2K)\cos.\theta]-\left(\frac{3\theta}{\text{sin. }\theta}-\frac{1}{\cos.^2\frac{1}{2}\theta}\right)\right\}$$

we must find the angle $\theta$ which corresponds to the maximum thrust by proceeding as in page 56. If it be less than $a$, or half the angle at the centre, this will be the angle of rupture, and the corresponding value of F or F′ will be the value of the horizontal thrust.

If the angle $\theta$, which answers to the maximum value

of F, is greater than $a$, we must put $a$ instead of $\theta$ in the above formula. We must find the maximum thrust due to sliding by the equation

$$G=\frac{r^2 \sin.\theta}{\tan.(\theta+\phi)}\left\{K^2(1-\tfrac{1}{2}\cos.\theta)-\tfrac{1}{3}\frac{\theta}{\sin.\theta}\right\},$$

$\phi$ being the angle of friction of the masonry. M. Petit takes $\phi=30°$.

The greatest value of F or G must be substituted in the equation

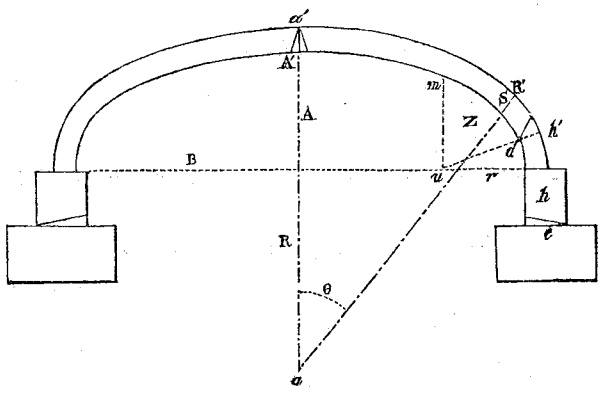
$(R+h-r\cos.a)\,e^2+(LR-\frac{1}{2}Lr\cos.a-r^2a)\,e+L^2(\frac{1}{4}R-\frac{1}{12}r\cos.a)-\frac{1}{2}Lr^2a+\frac{2}{3}Lr^2a+\frac{2}{3}r^3(1-\cos.a)=2F'(R+h-r\cos.a)$ in order to obtain $e$.

We need only observe that $\frac{F'}{r^2}=c$ or $F'=cr^2$, and that $c$ is given by the Tables.

We may proceed to the formulæ of Col. Audoy for the calculation of other arches.

*Formulæ relating to flat sweeps of three circular arcs with extrados parallel to the intrados.*

33. Let the angle A′$a$S (which subtends the $\frac{1}{2}$ arc at the summit)$=\theta$; let $a$ be the thickness of the arch, or A′ $a'$,




F

R the radius of the arc at the crown, $r$ the radii of the arcs at the springings, A the spring, or the height of the intrados under the key-stone, B the $\frac{1}{2}$ of the span, Z the angle $mud$ contained by the supposed joint of rupture $d\,h'$ and the vertical passing through $u$ the centre of the small arc at the springings, $h$ and $e$ the height and thickness of the abutments, S′ and Z′ the surfaces A′ S R′ $a'$, and S $d\,h'$ R′; N′ and L′ the moments of these surfaces taken with respect to the verticals A′ $a$ and $mu$; M′ M″ the moments of these surfaces taken with respect to the point $d$; F the value of the horizontal force; we must employ the formulæ

$$S'=\left\{\frac{(R+a)^2-R^2}{2}\right\}\theta,\ Z'=\left\{\frac{(r+a)^2-r^2}{2}\right\}(Z-\theta)$$

$$N'=\left\{\frac{(R+a)^3-R^3}{3}\right\}(1-\text{cos. }\theta)$$

$$L'=\left\{\frac{(r+a)^3-r^3}{3}\right\}(\text{cos. }\theta-\text{cos. Z})$$

$$M'=\left\{R-r\,(1-\text{sin. Z})\right\}S'-N';\ M''=r\,\text{sin. Z. Z}'-L',$$

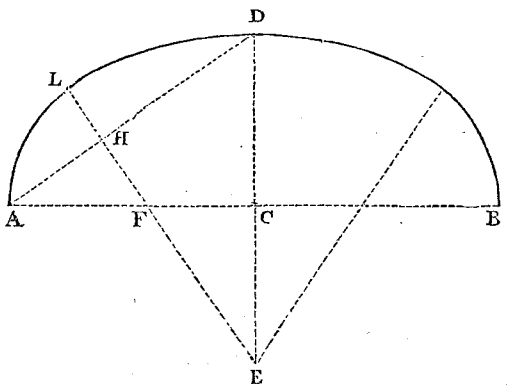
$$F=\frac{M'+M''}{A+a-r\,\text{cos. Z}}.$$

By taking for Z any angle greater than $\theta$, we may calculate all these values, and have the value of F corresponding to that angle. Similarly, we may find other values of F, by assuming different values of Z, and when we have obtained F′ the maximum value of F, we must substitute it in the equation $\frac{e^2h}{2}+e\,(S'+S'')+B\,S'+rS''-(N'+N'')=F'(A+a+h)$ (5). S″ is the surface of B′ SR′$b$, which is given by the equation $S''=\left\{\frac{(r+a)^2-r^2}{4}\right\}(\pi-2\,\theta)$; N″ is its moment with respect to the vertical

$mu$, and is given by the equation $N'' = \left\{\frac{(r+a)^3 - r^3}{3}\right\}$ cos. $\theta$.

For the methods of describing these arches, see "Papers on Bridges," pages 40 to 44.

When the arcs are 60°, and angle F E C =30°, then sin. 30°= ·5 and F C=F E sin. 30°=$\frac{1}{2}$ E F=$\frac{1}{2}$ (R−$r$) or R−$r$=2 F C, but $r$ = A C − F C, hence R=A C+F C.




Consequently we have in this case the radii; R=A C +F C and $r$=A C−F C respectively.

Let us give an example in which A C=30 feet or 10 yards, C D=4$\frac{1}{2}$ feet or 1$\frac{1}{2}$ yards, and the height of the pier=12 feet or 4 yards, also C D=$\frac{2}{3}$ A C, and since F E C =30°=$\theta$; sin. 30=·5; $\theta = \frac{30}{90} \cdot \frac{3{\cdot}1416}{2}$ = ·5236; B C =$\frac{2}{3}$ . 10=6$\frac{2}{3}$; and by "Papers on Bridges," (page 43,) F C=$\frac{\sqrt{3}+1}{2}$(B − A) = 4·55342; R = A C + F C = 14·55342; $r$=A C−F C=5·44658.

Substituting these values in the above formulæ, we have S′=12·01871; N′=47·11; Z′=9·295 (Z−·5236); L′=57·878 (·866−cos. Z); M″=5·44658 sin. Z$\left\{9{\cdot}295\right.$ (Z−·5236)$\left.\right\}$−57·878 (·866−cos. Z).

M′=$\left\{10-5{\cdot}44658\ (1-\text{sin. Z})\right\}$ 12·01871−47·11,

A+$a$−$r$ cos. Z=8·16666−5·44658 cos. Z.

Let $Z=45°$

$$Z=\frac{45}{90}.\frac{3\cdot1416}{2}=\cdot7854;\ \cos.Z=\sin.Z=\cdot7071;\ M'=53\cdot9$$

$$M''=\cdot166,\ A+a-r\cos.Z=4\cdot3154$$

$$F=\frac{M'+M''}{A+a-r\cos.Z.}=12\cdot5286.$$

If $Z=46°$,

$Z'=2\cdot5951640$, $L'=9\cdot897138$, $M''=\cdot269902$,
$M'=54\cdot6968742$, $A+a-r\cos.Z=4\cdot38$, $F=12\cdot54$.

If $Z=47°$,

$Z'=2\cdot7578265$, $L'=10\cdot651114706$, $M''=\cdot331474387$,
$M'=55\cdot49128068545732366$,
$A+a-r\cos.Z=4\cdot452156901194$, $F=12\cdot5383$.

Here the greatest value of F found is therefore $F=12\cdot54$ $=F'$, and corresponds to the angle of rupture $Z=46°$.

For the thickness of the piers in the case of strict equilibrium we must substitute the different terms in formula (5)

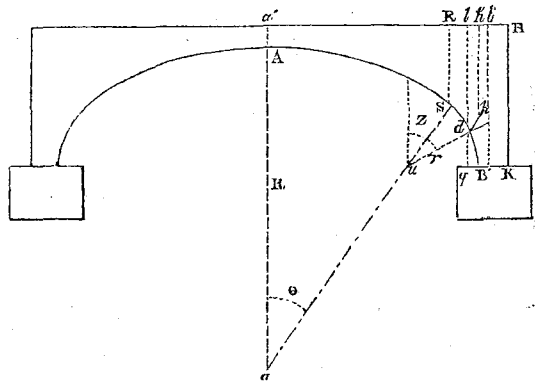
or $h=4$, $\theta=\cdot5236$, $\cos.\theta=\cdot866$, $\pi.=31416$; $F'=12\cdot54$ $a=1\cdot5$, $A=6\cdot666$, $S'=12\cdot01871$, $B=10$, $r=5\cdot44658$ $N'=47\cdot11$, $S''=9\cdot73372$, $N''=50\cdot12235$, then

$$e=-5\cdot4431-\sqrt{67\cdot92718}=2\cdot8.$$

To find the practical thickness we must increase the value of F' by $\frac{9}{10}$; before we introduce it in the formula (5), we should have substituted in that formula $12+54$ $+\frac{9}{10}\times12\cdot54=23\cdot83$ instead of $12\cdot54$, and the resulting value would be that which must be given to the thickness of the piers, in order that they may be able to resist any accidental causes.

*Formulæ relating to flat sweeps, described with three arcs of a circle with horizontal extrados.*

34. Let R be the radius of the arc at the crown, $r$ the radius of the arc at the springings, $e$ the thickness B′K at the springings, $a$ the thickness of the arch at the key-stone, $\theta$ the angle A $a$ $s$, Z the supposed angle of rupture, S′ and Z′ the surfaces A S R $a'$ and S $d$ $l$ R, N′ and L′ the moments of these surfaces with respect to the verticals A $a$ and $m$ $u$, M′ and M″ the moments of these surfaces with respect to the vertical $d$ $q$, M‴ the moment of the surface $d$ $h$ $h'$ $l$ with respect to the vertical $d$ $q$, F′ the maximum value of the thrust F. The formulæ are




$$S'=\frac{R\ \sin.\ \theta}{2}\left\{2\ (R+a)-R\cos.\ \theta\right\}-\frac{R^2\theta}{2};$$

$$N'=\frac{R^2\sin.^2\theta}{2}\ (R+a)-\frac{R^3}{3}\ (1-\cos.^3\theta);$$

$$Z'=r\ (A+a)\ (\sin.\ Z-\sin.\ \theta)-\frac{r^2}{2}\left\{Z+\sin.\ Z\cos.\ Z\right.$$
$$\left.-(\theta+\sin.\ \theta\cos.\ \theta)\right\};$$

$$L'=\frac{r^2}{2}\ (A+a)\ (\sin.^2 Z-\sin.^2\theta)-\frac{r^3}{3}(\cos.^3\theta-\cos.^3 Z)$$

$$M'=(B-r+r\sin.\ Z)\ S'-N';$$

$$M''=r\sin.\ Z\ .\ Z'-L';\quad M'''=a^2\sin.^2 Z$$
$$\left\{\frac{(A+a)-r\cos.\ Z}{2}-\frac{a\cos.\ Z}{3}\right\}$$

$F = \frac{M'+M''-M'''}{A+a-r\cos.\,Z}$. The equation which gives the thickness $e$ is $e^2\left(\frac{A+a}{2}\right) + P' + P'' = F\,(A+a)$ or $e\left(\frac{A+a}{2}\right)+e\,(S'+S'')+BS'+rS''-(N'+N'')=F(A+a)$

S″ represents the surface S B′ $b'$ R, which is found by the equation

$$S''=r\,(A+a)\,(1-\sin.\,\theta)-\frac{r^2}{2}\left(\frac{\pi-2\,\theta}{2}-\sin.\,\theta\cos.\,\theta\right);$$

N″ is its moment with respect to $m\,u$, which is

$$\frac{r^2}{2}(A+a)\,(1-\sin.^2\theta)-\frac{r^3\cos.^3\theta}{3};$$

P′ and P″ the moments of the surfaces A S R $a'$ and S B′ $b'$ R with respect to H $k$, which are found by the equations

$$P'=(e+B)\,S'-N';\quad P''=(e+r)\,S''-N''.$$

If the thickness of the piers of an arch $= e$, the heigh $= h$, the equation of equilibrium is

$$e^2\frac{(A+a+h)}{2}+e\,(S'+S'')+B\,S'+r\,S''-(N'+N'')$$
$$=F\,(A+a+h).$$

*Formulæ for Plate Bandes.*

35. In these arches the joints cannot be perpendicula to the intrados; they are so constructed that all th joints produced meet in the same point, (see page 29) Retaining the same notation as at page 28,

$$F=\frac{3\,a^2-t^2\,.\tan.^2\theta}{6},\text{ and}$$

$$e^2\left(\frac{h+t}{2}\right)+e\times t\times a+\frac{a^2\,t}{2}=F\,(a+h).$$

It is usual to construct an equilateral triangle upon the whole breadth, which finds 0 the common centre of all the joints. In this case we have $\theta = 30°$; $\tan.\theta = \frac{1}{\sqrt{3}}$ and $F = \frac{9a^2 - t^2}{18}$.

Let $a = 3{\cdot}5$ yards, $t = {\cdot}8$ yard, and $h = 3$ yards, and $F = \frac{9 \times (3{\cdot}5)^2 - ({\cdot}8)^2}{18} = 6{\cdot}089$, and $e^2 + 1{\cdot}47e = 9{\cdot}6$ $\therefore$ $e = 2{\cdot}445$ for the case of strict equilibrium. To obtain the practical thickness we must proceed as in page 68.

36. In concluding this part we may remark that the investigation of many other valuable formulæ is given in the Memoir of M.M. Lamé and Clapeyron, nearly the same as we have given from M.M. Petit and Audoy.

They determine the position of the point of rupture of the spherical extradossed dome of the church of St. Isaac at Petersburg, whose radii of intrados and extrados are 32 and 34 feet respectively, which give for the thickness of the dome the $\frac{1}{32}$ part of the interior diameter, and they find that this point is 68° 18′ from the key, or, which is the same, 21° 42′ from the springings.

In the supplement they give an investigation relative to circular cylindrical arches with horizontal extrados, as follows:

Retaining the same notation as at pages 43 and 44,

$$\frac{ma}{h} = \frac{3\,R\,r^2 \sin.^2\theta - r^3 \sin.^2\theta \cos.\theta - r^3\{3\theta \sin.\theta - 2(1 - \cos.\theta)\}}{6(R - r\cos.\theta)} \quad . \quad (a)$$

Differentiating the second member of this equation for $\theta$, and putting the result equal to nothing, making $\frac{R}{r} = K$

and $\frac{\theta}{\text{sin. }\theta} = z$, we have the maximum of $\frac{m\,a}{h}$,

$3z(1-\text{K}\cos.\theta)+2\cos.^3\theta-6\text{K}\cos.^2\theta+3(2\text{K}^2+1)\cos.\theta=3(\text{K}+2)$ (*b*

Calculating in the same manner as at page 46, w obtain the arc corresponding to the point of rupture fo each particular value of K or $\frac{\text{R}}{r}$, and this substituted i equation (*a*), will give the maximum value of $\frac{m\,a}{h}$.

They also deduce the following beautiful property, viz the point of rupture in an arch is that for which th tangent to the intrados at this point cuts the horizonta line passing through the summit of the key at the sam point as the vertical passing through the centre of gravit of the mass which tends to separate itself. The point o rupture being known, $\frac{m\,a}{h}$ is then equal to the mass *m* divided by the tangent of the angle which the lin touching the intrados at the point of rupture make with the horizon.

They further proceed to examine if there do no exist curves which give a constant moment of stability and find that this condition cannot be satisfied throughou the whole arch, but it may be so for a portion of the arch

For as $\frac{\text{M A}}{\text{H}}$ is constant in the same arch, they try i $\frac{m\,a}{h}$ can be so too, and find two different cases: in th first, the thickness of the arch must be = 0, in th second, the height of the key must = 0, neither of whic is admissible;—thus the moment of stability cannot b constant for the whole extent of the same arch.

TABLE I.—*Semicircular arches in which the extrados is parallel to the intrados.*

| Value of the ratio $K=\frac{R}{r}$ | Angle of rupture. | Ratio of the thrust C to the square of the radius $r$ of intrados. | | Ratio $\sqrt{2}\,c$ of the limit of the thickness of the piers to the radius of intrados. | |
|---|---|---|---|---|---|
| | | For the case of rotation. | For the case of sliding. | Strict equilibrium. | Equilibrium according to La Hire. |
| 2·732 | 0° 00′ | 0·00000 | 0·98923 | ,, | ,, |
| 2·70 | 13 42 | 0·00211 | 0·96262 | ,, | ,, |
| 2·65 | 22 00 | 0·00319 | 0·92168 | ,, | ,, |
| 2·60 | 27 30 | 0·00809 | 0·88151 | ,, | ,, |
| 2·50 | 35 52 | 0·02283 | 0·80346 | ,, | ,, |
| 2·40 | 42 6 | 0·04109 | 0·72847 | ,, | ,, |
| 2·30 | 46 47 | 0·06835 | 0·65654 | ,, | ,, |
| 2·20 | 51 4 | 0·08648 | 0·58767 | ,, | ,, |
| 2·10 | 54 27 | 0·10926 | 0·52186 | ,, | ,, |
| 2·00 | 57 17 | 0·13017 | 0·45912 | 0·9582 | 1·3223 |
| 1·90 | 59 37 | 0·14813 | 0·39943 | 0·8938 | 1·2320 |
| 1·80 | 61 24 | 0·16373 | 0·34281 | 0·8280 | 1·1414 |
| 1·70 | 62 53 | 0·17180 | 0·28924 | 0·7606 | 1·0484 |
| 1·60 | 63 49 | 0·17517 | 0·23874 | 0·6910 | 0·9525 |
| 1·59 | 63 52 | 0·17533 | 0·23386 | 0·6839 | 0·9427 |
| 1·58 | 63 55 | 0·17535 | 0·22901 | 0·6768 | 0·9329 |
| 1·57 | 63 58 | 0·17524 | 0·22434 | 0·6698 | 0·9233 |
| 1·56 | 64 1 | 0·17499 | 0·21940 | 0·6624 | 0·9131 |
| 1·55 | 64 3 | 0·17478 | 0·21464 | 0·6552 | 0·9031 |
| 1·54 | 64 5 | 0·17445 | 0·20991 | 0·6479 | 0·8931 |
| 1·53 | 64 7 | 0·17397 | 0·20521 | 0·6406 | 0·8831 |
| 1·52 | 64 8 | 0·17352 | 0·20054 | 0·6333 | 0·8730 |
| 1·51 | 64 8 | 0·17310 | 0·19590 | 0·6259 | 0·8628 |
| 1·50 | 64 9 | 0·17254 | 0·19130 | 0·6185 | 0·8527 |
| 1·49 | 64 8 | 0·17180 | 0·18673 | 0·6111 | 0·8424 |
| 1·48 | 64 8 | 0·17095 | 0·18218 | 0·6036 | 0·8320 |
| 1·47 | 64 7 | 0·17008 | 0·17766 | 0·5961 | 0·8216 |
| 1·46 | 64 6 | 0·16915 | 0·17318 | 0·5885 | 0·8112 |
| 1·45 | 64 5 | 0·16798 | 0·16872 | 0·5809 | 0·8007 |
| 1·44 | 64 3 | 0·16683 | 0·16430 | 0·5776 | 0·7962 |
| 1·43 | 64 00 | 0·16568 | 0·15991 | 0·5756 | 0·7934 |
| 1·42 | 63 56 | 0·16448 | 0·15555 | 0·5735 | 0·7906 |
| 1·41 | 63 52 | 0·16317 | 0·15122 | 0·5713 | 0·7874 |
| 1·40 | 63 48 | 0·16167 | 0·14691 | 0·5686 | 0·7838 |
| 1·39 | 63 43 | 0·16014 | 0·14264 | 0·5659 | 0·7801 |

| Value of the ratio $K=\frac{R}{r}$ | Angle of rupture. | Ratio of the thrust C to the square of the radius $r$ of intrados. | | Ratio $\sqrt{2}\,c$ of the limit of the thickness of the piers to the radius of intrados. | |
|---|---|---|---|---|---|
| | | For the case of rotation. | For the case of sliding. | Strict equilibrium. | Equilibrium according to La Hire. |
| 1·38 | 63° 38′ | 0·15845 | 0·13841 | 0·5629 | 0·7760 |
| 1·37 | 63 32 | 0·15672 | 0·13420 | 0·5598 | 0·7717 |
| 1·36 | 63 26 | 0·15482 | 0·13002 | 0·5564 | 0·7670 |
| 1·35 | 63 19 | 0·15287 | 0·12587 | 0·5529 | 0·7622 |
| 1·34 | 63 10 | 0·15096 | 0·12176 | 0·5495 | 0·7574 |
| 1·33 | 63 10 | 0·14896 | 0·11767 | 0·5458 | 0·7524 |
| 1·32 | 62 50 | 0·14678 | 0·11362 | 0·5418 | 0·7468 |
| 1·31 | 62 33 | 0·14510 | 0·10959 | 0·5387 | 0·7425 |
| 1·30 | 62 14 | 0·14330 | 0·10559 | 0·5353 | 0·7379 |
| 1·29 | 62 9 | 0·14013 | 0·10163 | 0·5294 | 0·7297 |
| 1·28 | 62 3 | 0·13691 | 0·09770 | 0·5233 | 0·7213 |
| 1·27 | 61 47 | 0·13430 | 0·09379 | 0·5183 | 0·7144 |
| 1·26 | 61 30 | 0·13157 | 0·08992 | 0·5130 | 0·7071 |
| 1·25 | 61 15 | 0·12847 | 0·08608 | 0·5069 | 0·6987 |
| 1·24 | 61 1 | 0·12516 | 0·08227 | 0·5003 | 0·6896 |
| 1·23 | 60 40 | 0·12201 | 0·07849 | 0·4940 | 0·6809 |
| 1·22 | 60 19 | 0·11887 | 0·07474 | 0·4876 | 0·6721 |
| 1·21 | 60 00 | 0·11516 | 0·07102 | 0·4799 | 0·6615 |
| 1·20 | 59 41 | 0·11140 | 0·06733 | 0·4720 | 0·6504 |
| 1·19 | 59 10 | 0·10791 | 0·06368 | 0·4646 | 0·6404 |
| 1·18 | 58 40 | 0·10417 | 0·06005 | 0·4564 | 0·6292 |
| 1·17 | 58 9 | 0·10021 | 0·05646 | 0·4472 | 0·6171 |
| 1·16 | 57 40 | 0·09593 | 0·05289 | 0·4380 | 0·6038 |
| 1·15 | 57 1 | 0·09176 | 0·04935 | 0·4284 | 0·5905 |
| 1·14 | 56 23 | 0·08729 | 0·04585 | 0·4178 | 0·5759 |
| 1·13 | 55 45 | 0·08254 | 0·04237 | 0·4063 | 0·5601 |
| 1·12 | 54 48 | 0·07789 | 0·03984 | 0·3947 | 0·5444 |
| 1·11 | 54 10 | 0·07273 | 0·03552 | 0·3814 | 0·5259 |
| 1·10 | 53 15 | 0·06754 | 0·03213 | 0·3675 | 0·5066 |
| 1·09 | 52 14 | 0·06177 | 0·02879 | ,, | ,, |
| 1·08 | 51 7 | 0·05649 | 0·02546 | ,, | ,, |
| 1·07 | 49 48 | 0·05065 | 0·02217 | ,, | ,, |
| 1·06 | 48 18 | 0·04455 | 0·01891 | ,, | ,, |
| 1·05 | 46 32 | 0·03813 | 0·01568 | ,, | ,, |
| 1·04 | 44 4 | 0·03139 | 0·01249 | ,, | ,, |
| 1·03 | 41 4 | 0·02459 | 0·00932 | ,, | ,, |
| 1·02 | 38 12 | 0·01691 | 0·00618 | ,, | ,, |
| 1·01 | 32 36 | 0·00889 | 0·00308 | ,, | ,, |
| 1·00 | 0 00 | 0·00000 | 0·00000 | ,, | ,, |

TABLE II.—*Semicircular arches with parallel extrados.—Table of the thickness of the piers.*

| Value of the ratio $K = \frac{R}{r}$ | Ratio $\frac{e}{r}$ of the thickness of the piers to the radius of intrados as a function of the ratio $\frac{r}{h}$ of radius to the height of the piers. (For strict equilibrium.) |
|---|---|
| 2·00 | $-2{\cdot}3562\frac{r}{h} + \sqrt{(5{\cdot}5517\frac{r^2}{h^2} + 1{\cdot}7907\frac{r}{h} + 0{\cdot}9182)}$ |
| 1·90 | $-2{\cdot}0449\frac{r}{h} + \sqrt{(4{\cdot}2021\frac{r^2}{h^2} + 1{\cdot}3240\frac{r}{h} + 0{\cdot}7988)}$ |
| 1·80 | $-1{\cdot}7593\frac{r}{h} + \sqrt{(3{\cdot}0951\frac{r^2}{h^2} + 0{\cdot}9368\frac{r}{h} + 0{\cdot}6856)}$ |
| 1·70 | $-1{\cdot}4844\frac{r}{h} + \sqrt{(2{\cdot}2034\frac{r^2}{h^2} + 0{\cdot}6933\frac{r}{h} + 0{\cdot}5785)}$ |
| 1·60 | $-1{\cdot}2252\frac{r}{h} + \sqrt{(1{\cdot}5012\frac{r^2}{h^2} + 0{\cdot}3775\frac{r}{h} + 0{\cdot}4775)}$ |
| 1·59 | $-1{\cdot}2001\frac{r}{h} + \sqrt{(1{\cdot}4404\frac{r^2}{h^2} + 0{\cdot}3566\frac{r}{h} + 0{\cdot}4677)}$ |
| 1·58 | $-1{\cdot}1752\frac{r}{h} + \sqrt{(1{\cdot}3812\frac{r^2}{h^2} + 0{\cdot}3361\frac{r}{h} + 0{\cdot}4580)}$ |
| 1·57 | $-1{\cdot}1513\frac{r}{h} + \sqrt{(1{\cdot}3255\frac{r^2}{h^2} + 0{\cdot}3151\frac{r}{h} + 0{\cdot}4487)}$ |
| 1·56 | $-1{\cdot}1261\frac{r}{h} + \sqrt{(1{\cdot}2677\frac{r^2}{h^2} + 0{\cdot}2966\frac{r}{h} + 0{\cdot}4388)}$ |
| 1·55 | $-1{\cdot}1015\frac{r}{h} + \sqrt{(1{\cdot}2133\frac{r^2}{h^2} + 0{\cdot}2783\frac{r}{h} + 0{\cdot}4293)}$ |
| 1·54 | $-1{\cdot}0772\frac{r}{h} + \sqrt{(1{\cdot}1605\frac{r^2}{h^2} + 0{\cdot}2603\frac{r}{h} + 0{\cdot}4198)}$ |
| 1·53 | $-1{\cdot}0531\frac{r}{h} + \sqrt{(1{\cdot}1091\frac{r^2}{h^2} + 0{\cdot}2428\frac{r}{h} + 0{\cdot}4104)}$ |
| 1·52 | $-1{\cdot}0292\frac{r}{h} + \sqrt{(1{\cdot}0592\frac{r^2}{h^2} + 0{\cdot}2224\frac{r}{h} + 0{\cdot}4011)}$ |
| 1·51 | $-1{\cdot}0073\frac{r}{h} + \sqrt{(1{\cdot}0146\frac{r^2}{h^2} + 0{\cdot}2056\frac{r}{h} + 0{\cdot}3918)}$ |
| 1·50 | $-0{\cdot}9817\frac{r}{h} + \sqrt{(0{\cdot}9638\frac{r^2}{h^2} + 0{\cdot}1937\frac{r}{h} + 0{\cdot}3826)}$ |
| 1·49 | $-0{\cdot}9583\frac{r}{h} + \sqrt{(0{\cdot}9184\frac{r^2}{h^2} + 0{\cdot}1684\frac{r}{h} + 0{\cdot}3735)}$ |
| 1·48 | $-0{\cdot}9349\frac{r}{h} + \sqrt{(0{\cdot}8741\frac{r^2}{h^2} + 0{\cdot}1659\frac{r}{h} + 0{\cdot}3644)}$ |

| Value of the ratio $K=\frac{R}{r}$ | Ratio $\frac{e}{r}$ of the thickness of the piers to the radius of intrados as a function of the ratio $\frac{r}{h}$ of radius to the height of the piers. (For strict equilibrium.) |
|---|---|
| 1·47 | $-0{\cdot}9125\frac{r}{h}+\sqrt{(0{\cdot}8328\frac{r^2}{h^2}+0{\cdot}1482\frac{r}{h}+0{\cdot}3553)}$ |
| 1·46 | $-0{\cdot}8887\frac{r}{h}+\sqrt{(0{\cdot}7899\frac{r^2}{h^2}+0{\cdot}1362\frac{r}{h}+0{\cdot}3464)}$ |
| 1·45 | $-0{\cdot}8659\frac{r}{h}+\sqrt{(0{\cdot}7498\frac{r^2}{h^2}+0{\cdot}1232\frac{r}{h}+0{\cdot}3374)}$ |
| 1·44 | $-0{\cdot}8432\frac{r}{h}+\sqrt{(0{\cdot}7110\frac{r^2}{h^2}+0{\cdot}1181\frac{r}{h}+0{\cdot}3337)}$ |
| 1·43 | $-0{\cdot}8206\frac{r}{h}+\sqrt{(0{\cdot}6735\frac{r^2}{h^2}+0{\cdot}1153\frac{r}{h}+0{\cdot}3314)}$ |
| 1·42 | $-0{\cdot}7983\frac{r}{h}+\sqrt{(0{\cdot}6372\frac{r^2}{h^2}+0{\cdot}1143\frac{r}{h}+0{\cdot}3290)}$ |
| 1·41 | $-0{\cdot}7760\frac{r}{h}+\sqrt{(0{\cdot}6023\frac{r^2}{h^2}+0{\cdot}1102\frac{r}{h}+0{\cdot}3263)}$ |
| 1·40 | $-0{\cdot}7540\frac{r}{h}+\sqrt{(0{\cdot}5685\frac{r^2}{h^2}+0{\cdot}1074\frac{r}{h}+0{\cdot}3233)}$ |
| 1·39 | $-0{\cdot}7321\frac{r}{h}+\sqrt{(0{\cdot}5359\frac{r^2}{h^2}+0{\cdot}1048\frac{r}{h}+0{\cdot}3203)}$ |
| 1·38 | $-0{\cdot}7103\frac{r}{h}+\sqrt{(0{\cdot}5045\frac{r^2}{h^2}+0{\cdot}1021\frac{r}{h}+0{\cdot}3169)}$ |
| 1·37 | $-0{\cdot}6887\frac{r}{h}+\sqrt{(0{\cdot}4743\frac{r^2}{h^2}+0{\cdot}0995\frac{r}{h}+0{\cdot}3134)}$ |
| 1·36 | $-0{\cdot}6673\frac{r}{h}+\sqrt{(0{\cdot}4452\frac{r^2}{h^2}+0{\cdot}0969\frac{r}{h}+0{\cdot}3096)}$ |
| 1·35 | $-0{\cdot}6460\frac{r}{h}+\sqrt{(0{\cdot}4173\frac{r^2}{h^2}+0{\cdot}0944\frac{r}{h}+0{\cdot}3057)}$ |
| 1·34 | $-0{\cdot}6249\frac{r}{h}+\sqrt{(0{\cdot}3904\frac{r^2}{h^2}+0{\cdot}0926\frac{r}{h}+0{\cdot}3019)}$ |
| 1·33 | $-0{\cdot}6050\frac{r}{h}+\sqrt{(0{\cdot}3660\frac{r^2}{h^2}+0{\cdot}0903\frac{r}{h}+0{\cdot}2979)}$ |
| 1·32 | $-0{\cdot}5831\frac{r}{h}+\sqrt{(0{\cdot}3400\frac{r^2}{h^2}+0{\cdot}0880\frac{r}{h}+0{\cdot}2936)}$ |
| 1·31 | $-0{\cdot}5624\frac{r}{h}+\sqrt{(0{\cdot}3163\frac{r^2}{h^2}+0{\cdot}0875\frac{r}{h}+0{\cdot}2902)}$ |
| 1·30 | $-0{\cdot}5419\frac{r}{h}+\sqrt{(0{\cdot}2937\frac{r^2}{h^2}+0{\cdot}0867\frac{r}{h}+0{\cdot}2866)}$ |
| 1·29 | $-0{\cdot}5216\frac{r}{h}+\sqrt{(0{\cdot}2720\frac{r^2}{h^2}+0{\cdot}0828\frac{r}{h}+0{\cdot}2803)}$ |

| Value of the ratio $K=\frac{R}{r}$ | Ratio $\frac{e}{r}$ of the thickness of the piers to the radius of intrados as a function of the ratio $\frac{r}{h}$ of radius to the height of the piers. (For strict equilibrium.) |
|---|---|
| 1·28 | $-0{\cdot}5014\frac{r}{h}+\surd(0{\cdot}2520\frac{r^2}{h^2}+0{\cdot}0801\frac{r}{h}+0{\cdot}2738)$ |
| 1·27 | $-0{\cdot}4926\frac{r}{h}+\surd(0{\cdot}2426\frac{r^2}{h^2}+0{\cdot}0778\frac{r}{h}+0{\cdot}2686)$ |
| 1·26 | $-0{\cdot}4615\frac{r}{h}+\surd(0{\cdot}2130\frac{r^2}{h^2}+0{\cdot}0755\frac{r}{h}+0{\cdot}2631)$ |
| 1·25 | $-0{\cdot}4418\frac{r}{h}+\surd(0{\cdot}1952\frac{r^2}{h^2}+0{\cdot}0730\frac{r}{h}+0{\cdot}2569)$ |
| 1·24 | $-0{\cdot}4222\frac{r}{h}+\surd(0{\cdot}1783\frac{r^2}{h^2}+0{\cdot}0713\frac{r}{h}+0{\cdot}2503)$ |
| 1·23 | $-0{\cdot}4028\frac{r}{h}+\surd(0{\cdot}1623\frac{r^2}{h^2}+0{\cdot}0684\frac{r}{h}+0{\cdot}2440)$ |
| 1·22 | $-0{\cdot}3836\frac{r}{h}+\surd(0{\cdot}1471\frac{r^2}{h^2}+0{\cdot}0674\frac{r}{h}+0{\cdot}2377)$ |
| 1·21 | $-0{\cdot}3645\frac{r}{h}+\surd(0{\cdot}1329\frac{r^2}{h^2}+0{\cdot}0641\frac{r}{h}+0{\cdot}2303)$ |
| 1·20 | $-0{\cdot}3456\frac{r}{h}+\surd(0{\cdot}1194\frac{r^2}{h^2}+0{\cdot}0614\frac{r}{h}+0{\cdot}2228)$ |
| 1·19 | $-0{\cdot}3268\frac{r}{h}+\surd(0{\cdot}1068\frac{r^2}{h^2}+0{\cdot}0600\frac{r}{h}+0{\cdot}2158)$ |
| 1·18 | $-0{\cdot}3082\frac{r}{h}+\surd(0{\cdot}0950\frac{r^2}{h^2}+0{\cdot}0581\frac{r}{h}+0{\cdot}2083)$ |
| 1·17 | $-0{\cdot}2897\frac{r}{h}+\surd(0{\cdot}0840\frac{r^2}{h^2}+0{\cdot}0561\frac{r}{h}+0{\cdot}2004)$ |
| 1·16 | $-0{\cdot}2714\frac{r}{h}+\surd(0{\cdot}0734\frac{r^2}{h^2}+0{\cdot}0559\frac{r}{h}+0{\cdot}1919)$ |
| 1·15 | $-0{\cdot}2533\frac{r}{h}+\surd(0{\cdot}0642\frac{r^2}{h^2}+0{\cdot}0536\frac{r}{h}+0{\cdot}1835)$ |
| 1·14 | $-0{\cdot}2353\frac{r}{h}+\surd(0{\cdot}0554\frac{r^2}{h^2}+0{\cdot}0513\frac{r}{h}+0{\cdot}1745)$ |
| 1·13 | $-0{\cdot}2175\frac{r}{h}+\surd(0{\cdot}0473\frac{r^2}{h^2}+0{\cdot}0490\frac{r}{h}+0{\cdot}1651)$ |
| 1·12 | $-0{\cdot}1998\frac{r}{h}+\surd(0{\cdot}0399\frac{r^2}{h^2}+0{\cdot}0467\frac{r}{h}+0{\cdot}1557)$ |
| 1·11 | $-0{\cdot}1823\frac{r}{h}+\surd(0{\cdot}0332\frac{r^2}{h^2}+0{\cdot}0426\frac{r}{h}+0{\cdot}1455)$ |
| 1·10 | $-0{\cdot}1649\frac{r}{h}+\surd(0{\cdot}0272\frac{r^2}{h^2}+0{\cdot}0394\frac{r}{h}+0{\cdot}1351)$ |

TABLE III.—*Semicircular arches in which the extrados is horizontal.—Table of the angles of rupture, of the thrusts, and the limit of the thickness of the piers.*

| Value of the ratio $K=\frac{R}{r}$ | Angle of rupture. | Ratio C of the thrust to the square of the radius $r$ of intrados. | | Ratio of the limit of the thickness of the piers to the radius $r$ of intrados. | |
|---|---|---|---|---|---|
| | | For rotation. | For sliding. | Strict equilibrium. | Stability according to La Hire. |
| 2·00 | 36° | 0·05486 | 0·50358 | 1·0036 | 1·3834 |
| 1·90 | 39 | 0·07101 | 0·43966 | 0·9377 | 1·2925 |
| 1·80 | 44 | 0·08850 | 0·37901 | 0·8706 | 1·2001 |
| 1·70 | 48 | 0·10631 | 0·32164 | 0·8020 | 1·1055 |
| 1·60 | 52 | 0·12300 | 0·26755 | 0·7315 | 1·0082 |
| 1·59 | 52 | 0·12453 | 0·26232 | 0·7243 | 0·9984 |
| 1·58 | 53 | 0·12602 | 0·25712 | 0·7171 | 0·9885 |
| 1·57 | 53 | 0·12747 | 0·25196 | 0·7099 | 0·9784 |
| 1·56 | 54 | 0·12837 | 0·24683 | 0·7026 | 0·9684 |
| 1·55 | 54 | 0·13027 | 0·24173 | 0·6953 | 0·9584 |
| 1·54 | 55 | 0·13153 | 0·23667 | 0·6880 | 0·9483 |
| 1·53 | 55 | 0·13289 | 0·23163 | 0·6806 | 0·9381 |
| 1·52 | 55 | 0·13414 | 0·22664 | 0·6732 | 0·9280 |
| 1·51 | 55 | 0·13531 | 0·22167 | 0·6658 | 0·9177 |
| 1·50 | 56 | 0·13648 | 0·21673 | 0·6583 | 0·9075 |
| 1·49 | 56 | 0·13756 | 0·21183 | 0·6509 | 0·8972 |
| 1·48 | 56 | 0·13856 | 0·20696 | 0·6433 | 0·8868 |
| 1·47 | 57 | 0·13952 | 0·20213 | 0·6358 | 0·8764 |
| 1·46 | 57 | 0·14041 | 0·19733 | 0·6282 | 0·8659 |
| 1·45 | 57 | 0·14122 | 0·19256 | 0·6206 | 0·8554 |
| 1·44 | 58 | 0·14195 | 0·18782 | 0·6129 | 0·8448 |
| 1·43 | 58 | 0·14268 | 0·18312 | 0·6052 | 0·8341 |
| 1·42 | 58 | 0·14311 | 0·17845 | 0·5974 | 0·8234 |
| 1·41 | 59 | 0·14376 | 0·17381 | 0·5896 | 0·8126 |
| 1·40 | 59 | 0·14421 | 0·16920 | 0·5817 | 0·8018 |
| 1·39 | 59 | 0·14456 | 0·16463 | 0·5738 | 0·7909 |
| 1·38 | 59 | 0·14481 | 0·16009 | 0·5658 | 0·7799 |
| 1·37 | 60 | 0·14498 | 0·15558 | 0·5578 | 0·7689 |
| 1·36 | 60 | 0·14506 | 0·15111 | 0·5497 | 0·7577 |

| Value of the ratio $K = \frac{R}{r}$ | Angle of rupture. | Ratio C of the thrust to the square of the radius $r$ of intrados. | | Ratio of the limit of the thickness of the piers to the radius $r$ of intrados. | |
|---|---|---|---|---|---|
| | | For rotation. | For sliding. | Strict equilibrium. | Stability according to La Hire. |
| 1·35 | 60° | 0·14504 | 0·14666 | 0·5416 | 0·7465 |
| 1·34 | 60 | 0·14491 | 0·14225 | 0·5383 | 0·7420 |
| 1·33 | 61 | 0·14467 | ,, | 0·5379 | 0·7414 |
| 1·32 | 61 | 0·14460 | ,, | 0·5377 | 0·7412 |
| 1·31 | 61 | 0·14390 | ,, | 0·5358 | 0·7394 |
| 1·30 | 61 | 0·14322 | 0·12495 | 0·5354 | 0·7379 |
| 1·29 | 61 | 0·14264 | ,, | 0·5341 | 0·7362 |
| 1·28 | 62 | 0·14186 | ,, | 0·5326 | 0·7342 |
| 1·27 | 62 | 0·14101 | ,, | 0·5310 | 0·7320 |
| 1·26 | 62 | 0·13988 | ,, | 0·5289 | 0·7290 |
| 1·25 | 62 | 0·13872 | 0·10405 | 0·5267 | 0·7260 |
| 1·24 | 62 | 0·13737 | ,, | 0·5235 | 0·7225 |
| 1·23 | 63 | 0·13593 | ,, | 0·5214 | 0·7187 |
| 1·22 | 63 | 0·13437 | ,, | 0·5184 | 0·7145 |
| 1·21 | 63 | 0·13263 | ,, | 0·5150 | 0·7099 |
| 1·20 | 63 | 0·13073 | 0·08397 | 0·5113 | 0·7048 |
| 1·19 | 63 | 0·12870 | ,, | 0·5073 | 0·6993 |
| 1·18 | 63 | 0·12650 | ,, | 0·5030 | 0·6933 |
| 1·17 | 64 | 0·12415 | ,, | 0·4983 | 0·6868 |
| 1·16 | 64 | 0·12182 | ,, | 0·4936 | 0·6803 |
| 1·15 | 64 | 0·11895 | 0·06471 | 0·4877 | 0·6723 |
| 1·14 | 64 | 0·11608 | ,, | 0·4818 | 0·6641 |
| 1·13 | 64 | 0·11303 | ,, | 0·4755 | 0·6553 |
| 1·12 | 64 | 0·10979 | ,, | 0·4886 | 0·6459 |
| 1·11 | 65 | 0·10641 | ,, | 0·4613 | 0·6358 |
| 1·10 | 65 | 0·10279 | 0·04627 | 0·4535 | 0·6249 |
| 1·09 | 66 | 0·098992 | ,, | 0·4449 | 0·6133 |
| 1·08 | 66 | 0·094967 | ,, | 0·4358 | 0·6007 |
| 1·07 | 67 | 0·091189 | ,, | 0·4270 | 0·5886 |
| 1·06 | 68 | 0·086376 | ,, | 0·4156 | 0·5729 |
| 1·05 | 69 | 0·081755 | 0·02865 | 0·4044 | 0·5573 |
| 1·04 | 70 | 0·076857 | ,, | ,, | ,, |
| 1·03 | 71 | 0·071853 | ,, | ,, | ,, |
| 1·02 | 73 | 0·066469 | ,, | ,, | ,, |
| 1·01 | 74 | 0·061324 | ,, | ,, | ,, |
| 1·00 | 75 | 0·055472 | 0·01185 | ,, | ,, |

TABLE IV.—*Arches in the form of an arc of a circle with parallel ext*
*Tables of thrusts in various systems.*

| Value of the ratio $K=\frac{R}{r}$ | RATIO OF THE THRUST TO THE SQUARE OF THE RADIUS OF EXTR | | | | | |
|---|---|---|---|---|---|---|
| | System where $L=4f$ $r=\frac{5}{2}f$ $a=53°7'30''$ | System where $L=5f$ $r=\frac{29}{8}f$ $a=43°36'10''$ | System where $L=6f$ $r=5f$ $a=36°52'10''$ | System where $L=7f$ $r=\frac{53}{8}f$ $a=31°53'26''$ | System where $L=8f$ $r=\frac{17}{2}f$ $a=28°4'20''$ | System $L=$ $r=$ $a=22°$ |
| 1·40 | 0·15445 | 0·14691 | 0·14691 | 0·14691 | 0·14691 | 0·14 |
| 1·35 | 0·14717 | 0·13030 | 0·12587 | 0·12587 | 0·12587 | 0·12 |
| 1·34 | 0·14543 | 0·12987 | 0·12171 | 0·12171 | 0·12171 | 0·11 |
| 1·33 | 0·14364 | 0·12781 | 0·11767 | 0·11767 | 0·11767 | 0·11 |
| 1·32 | 0·14173 | 0·12634 | 0·11362 | 0·11362 | 0·11362 | 0·11 |
| 1·31 | 0·13975 | 0·12486 | 0·10959 | 0·10959 | 0·10959 | 0·10 |
| 1·30 | 0·13764 | 0·12331 | 0·10682 | 0·10559 | 0·10559 | 0·10 |
| 1·29 | 0·13543 | 0·12164 | 0·10563 | 0·10163 | 0·10163 | 0·10 |
| 1·28 | 0·13311 | 0·11988 | 0·10437 | 0·09770 | 0·09770 | 0·09 |
| 1·27 | 0·13068 | 0·11803 | 0·10304 | 0·09379 | 0·09379 | 0·09 |
| 1·26 | 0·12815 | 0·11609 | 0·10160 | 0·08992 | 0·08992 | 0·08 |
| 1·25 | 0·12547 | 0·11402 | 0·10009 | 0·08668 | 0·08608 | 0·08 |
| 1·24 | 0·12270 | 0·11251 | 0·09850 | 0·08549 | 0·08227 | 0·08 |
| 1·23 | 0·12031 | 0·10958 | 0·09679 | 0·08423 | 0·07849 | 0·07 |
| 1·22 | 0·11675 | 0·10725 | 0·09499 | 0·08291 | 0·07474 | 0·07 |
| 1·21 | 0·11354 | 0·10460 | 0·09305 | 0·08148 | 0·07102 | 0·06 |
| 1·20 | 0·11023 | 0·10196 | 0·09102 | 0·07999 | 0·06981 | 0·06 |
| 1·19 | 0·10676 | 0·09915 | 0·08885 | 0·07834 | 0·06859 | 0·06 |
| 1·18 | 0·10313 | 0·09617 | 0·08653 | 0·07651 | 0·06727 | 0·06 |
| 1·17 | 0·09934 | 0·09303 | 0·08408 | 0·07468 | 0·06583 | 0·05 |
| 1·16 | 0·09537 | 0·08975 | 0·08144 | 0·07264 | 0·06420 | 0·05 |
| 1·15 | 0·09123 | 0·08634 | 0·07866 | 0·07050 | 0·06259 | 0·04 |
| 1·14 | 0·08690 | 0·08257 | 0·07568 | 0·06812 | 0·06077 | 0·04 |
| 1·13 | 0·08238 | 0·07869 | 0·07251 | 0·06558 | 0·05890 | 0·04 |
| 1·12 | 0·07764 | 0·07459 | 0·06911 | 0·06297 | 0·05659 | 0·04 |
| 1·11 | 0·07269 | 0·07042 | 0·06548 | 0·06026 | 0·05421 | 0·04 |
| 1·10 | 0·06737 | 0·06563 | 0·06158 | 0·05666 | 0·05160 | 0·04 |
| 1·09 | 0·06211 | 0·06077 | 0·05739 | 0·05345 | 0·04871 | 0·04 |
| 1·08 | 0·05636 | 0·05652 | 0·05288 | 0·04934 | 0·04552 | 0·03 |
| 1·07 | 0·05052 | 0·05011 | 0·04804 | 0·04426 | 0·04200 | 0·03 |
| 1·06 | 0·04431 | 0·04428 | 0·04280 | 0·04058 | 0·03861 | 0·03 |
| 1·05 | 0·03776 | 0·03804 | 0·03709 | 0·03550 | 0·03357 | 0·03 |
| 1·04 | 0·03096 | 0·03144 | 0·03095 | 0·02992 | 0·02862 | 0·02 |
| 1·03 | 0·02378 | 0·02437 | 0·02424 | 0·02369 | 0·02293 | 0·02 |
| 1·02 | 0·01625 | 0·01681 | 0·01690 | 0·01673 | 0·01640 | 0·01 |
| 1·01 | 0·00834 | 0·00871 | 0·00886 | 0·00889 | 0·00885 | 0·00 |

# PAPERS ON BRIDGES.

GAUTHEY.

## I. ON THE GENERAL PRINCIPLES THAT OUGHT TO REGULATE THE CONSTRUCTION OF BRIDGES, AND DETERMINE THE DIMENSIONS OF THEIR SEVERAL PARTS.

THE principal object to be observed in forming the plan of a bridge, is to give a suitable and convenient aperture to the arches, so as to afford a free vent to the waters of sudden floods or inundations, and to secure the solidity and duration of the edifice by a skilful construction.

The solidity of a bridge depends almost entirely on the manner in which its foundations are laid. When these are once properly arranged, the upper part may be erected either with simplicity or elegance, without impairing in any degree the durability of the structure. Experience has proved that many bridges either decay, or are swept away by sudden floods, by reason of the defective mode of fixing their foundations, while very few suffer from an unskilful construction of the piles or arches. This latter defect, however, is easy of cor-

rection, nor is it difficult to prevent the consequences that might be expected from it.

It is to be observed, that almost all the bridges erected before the eighteenth century are built with a view to economy, both with respect to the style of building, and the degree of breadth allotted to them, since the most considerable scarcely allow room for two carriages to pass abreast. The bridges of Paris are indeed, for the most part, very wide; but this extent has been given to them solely with a view to erect two rows of small houses on their sides—a circumstance that must have naturally produced a very narrow thoroughfare.

In most of the old bridges in France, the arches are not large; and, with the exception of the projections and angles of the walls, which are of free-stone, the rest of the building consists of a rough kind of stone, the cost of which is much less; yet these bridges are, however, durable and lasting. In the last century more attention was paid to elegance, and to the display of the taste and ingenuity of the architect. Even at a distance from cities or large towns, spacious bridges were erected, and raised on very long and deep sunk arches, the bold and difficult construction of which required free-stone of great size and extraordinary price. The consequence has been, that these beautiful bridges reflected honour on France, and inspired foreigners with a high idea of the perfection to which the art had been carried; but the small number of the grand bridges that arose from this system has completely absorbed the resources that the government devoted to this department, and obliged it to neglect

bridges that were on roads of greater importance, and vicinities of superior population, which, however, would have been of immense utility to the general interests of commerce.

It is an indispensable consideration in these matters, to make a distinction between the different species of bridges with respect to their size, and the mode of their construction. Such as are built on roads of the second or third class, and in towns of no great extent, ought not to be formed on the same plan as the bridges over roads more frequented by foreigners, or situated at the entry, or in the interior of extensive cities. In the former class, solidity and durability ought to be kept in view; in the latter, something more is required with respect to elegance and skill.

By the adoption of these principles, the real interests of the government will be secured, and its resources will not be squandered on useless projects.

In the projection of a bridge five principal points are necessary to be considered: 1st, the choice of its position or locality; 2nd, the vent, or egress that must be allowed to the river; 3rd, the form of the arches; 4th, the size of the arches; 5th, the breadth of the bridge. Certain rules arising from the local circumstances determine the nature of these several considerations; and in the concocting of the plan nothing should ever be left to chance or caprice.

When these fundamental positions are thus laid down and determined, nothing remains but to fix the particular dimensions of each part, the solidity of the arches, and

that of the piles and buttresses; and to choose the mode of foundation best to be observed, and the nature of the materials intended for the construction of the bridge. It will, however, often happen that these latter considerations will operate on the previous determinations; and it will be necessary, in order to come to a definitive resolution, to combine together the different circumstances of locality, and have them all present to the mind at once.

An exact knowledge of the ground or locality is indispensably necessary in the formation of the plan. It will be requisite to have a sketch of the course of the river, sufficiently extensive to furnish an idea of its general nature, the changes that its bed may have experienced, or such as may subsequently occur. In cases where the position of the bridge is not previously adopted, the plan should be furnished with certain data that would lead to a selection of this important point. It is equally necessary to ascertain the level of the stream, in order to discover its slope; and this level ought to be taken during the different seasons of the year, that a due judgment may be formed respecting the variations that floods might occasion, either in the slope or declivity, or in the mode of its distribution.

In addition to the various sketches that may be drawn up descriptive of the length of the river's course, others should be taken across its bed, so as to determine its breadth, its form, and the depth of its waters at the different periods of the year; and it is especially essential to establish two fixed points, the one relative to the

lowest, and the other to the highest, elevations of the water which have been observed. These elucidations will be useful to calculate the different heights of the foundations of the various parts of the bridge, and those of the arches, as well as the steps that must be formed at the landing-places. The measure of the velocity of the water should also be annexed to these documents, and this is particularly important at the period of sudden floods.

The plan and sections of the course of the stream, and the measure of its velocity, will serve to determine the position of the bridge, and the principal dimensions of its parts. But in order to ascertain the best method of forming its foundation, the nature of the soil on which its waters flow must be investigated, which knowledge will be derived from accurate soundings taken at different parts, and at proper depths, that will lead to an accurate conclusion on this point. When we come to treat of the Foundation of Bridges, the proper mode of proceeding with respect to this operation, as well as other necessary points of information on the same subject, will be given.

It will likewise be essentially requisite to procure exact authority on the nature and value of the materials to be employed, on the possibility of collecting a sufficient number of workmen during the different seasons of the year, and on the degree of talent and skill that they may possess. The architect ought carefully to collect all these materials of information, since they constitute the primary elements of the plan which he means to pursue.

# ON THE CONSTRUCTION OF BRIDGES.

## II. ON THE POSITION OF BRIDGES.

THE spot on which a bridge is to be built is not always at the disposal of the constructor. In the country it is usually determined by the direction of the roads, and in the interior of cities by the position of the streets, or their immediate vicinities. Nothing then is left to the constructor but to overcome the obstacles which nature opposes to him in the quarter which is consigned to his care.

It must be acknowledged, however, that the choice of this position is sometimes arbitrary, and even in cities, where it is intended to open new streets, especially when the old ones are too narrow and intricate. This method has been adopted at Mantes, Orleans, Moulins, &c. In such a case it will be proper to erect the bridge on the most solid ground, on one not apt to shrink, or be wasted by the force of the current. Rock being always the best basis for foundations, it may be allowable sometimes to deviate in search of it when it does not lie too deep, and to lengthen the road, rather than expose the solidity and duration of the work by constructing it on

an unsafe soil. But it is clear that it is impossible to lay down general rules on this point, since the matter is generally decided by the particular circumstances of the case; and we need only point out the principal motives by which the position of the bridge ought to be regulated and determined.

It will be essentially necessary to place the axis of the bridge perpendicular to the level of the water, so that the direction of the current shall be parallel to the lateral faces of the piles. When this is not possible, the faces of the piles are inclined relatively to the axis of the bridge, which then takes the name of oblique or sloping bridge. A construction of this kind must be adopted when the road makes an angle different from a right one with the course of the river.

This species of bridge is generally avoided, principally when it has many arches, on account of the difficulty of this construction; but this difficulty, which is dependent on mere matters of detail, is far from deserving the importance usually assigned to it. No reluctance should ever be felt to build a bridge of this stamp, where any inconvenience or irregularity can be removed, either in the direction of the road, or the ordinary current of the stream.

### ON THE DISCHARGE OF THE WATERS FROM BRIDGES.

This subject is one of great importance in the construction of bridges, as it operates on their durability in proportion as its due solution is either more or less

exact. This problem, however, is unfortunately too like all those which are connected with the physico-mathematical sciences, as we are obliged, in order to resolve it, to have recourse to defective data, which always lead to uncertainty in the results obtained from them, until numerous and authenticated experiments shall have removed every obscurity on the subject.

The discharge or vent that ought to be allotted to a river, is less difficult to settle when there are bridges in its vicinity, built on the same stream. In that case it will be only necessary to measure, during its highest elevation, the section of the river at the passages of those bridges, and to observe the velocity of the water, and the fall that usually arises from the upper part of the stream.

By means of conclusions drawn from these data, we may be enabled to fix the vent or discharge in a manner sufficiently exact. But if no bridge exists either above or below the intended construction, we must only refer, in order to solve this problem, to the rules which we shall lay down, and which may be usefully adopted in all cases, even though they should present nothing more than a medium of verification. The question of the discharges naturally involves two other points of inquiry—1st, to determine, after a knowledge of the bed of the river, what quantity of water the bridge should let pass; 2nd, this quantity of water being ascertained, to fix the surface or extent of the necessary discharge.

*On the manner of calculating the quantity of water to which the bridge should afford a free passage.*

It is evident that the volume of water discharged by any river is not the same in all the seasons of the year. Not only is it less considerable in summer than in winter, but all rivers are subject to occasional increases produced by temporary and profuse showers of rain, or by the melting of snow or ice. The discharge of a river ought therefore to be extensive enough to allow not only a passage to the mean quantity of water in the bed of the river, but also to afford a vent to the surplus quantity of water that arises from floods or inundations; and it is particularly, by paying attention to this latter circumstance, that the proper dimensions of the arches can be ascertained.

This quantity of water seems at first to preserve a regular proportion to the surface of the ground over which it runs, and approaches the point where the intended bridge is to be erected. This supposition, however, will lead to very serious errors. In fact, it has been observed that the quantity of water that falls during the same year is very different in different places; and moreover, the nature and the inclinations of the ground that receives it, has much influence on the manner in which it discharges itself with more or less velocity, or penetrates the earth to a greater or lesser depth. Besides, it is easy to see that it is not so much the quantity of water that falls during the whole year, as that which proceeds from heavy rains, or the melting of snow, that

ought to be taken into calculation, as these produce frequently an extraordinary increase. However, if in many respects this consideration does not merit much attention, it ought not to be entirely neglected; it may afford room for many useful speculations, that may be applied to places in the vicinity of each other, and where the inclination and nature of the ground may be nearly similar.

There is another circumstance well deserving of attention, which is, that of the time in which the surplus water arising from a flood takes to discharge itself along the bed of the river, or the velocity with which this discharge is made. It is well known that this degree of velocity depends, in a great measure, on the slope of the river; and as this slope diminishes regularly in proportion as it recedes from its source, it follows that the same mass of water which flows rapidly in the vicinity of the mountains where the river takes its rise, and where it is still a torrent, will proceed with greater slowness during the rest of its course, as it advances towards the sea, or to the river into which it is discharged. Thus, admitting that it does not receive any considerable afflux, and that its bed is generally level, if two bridges are built on this same river, it will be necessary to give to the bridge that shall be nearer to the fountain-head a wider extent of discharge than to the other; since they must both afford a vent to the same increase of stream, which will take two days to clear off in the former case, and eight days in the latter.

It is laid down as a rule, that in order to calculate the

quantity of water that flows in a river, it will be necessary to multiply the surface of the section by the mean velocity of the current. The first of these elements of calculation is always easy to be ascertained with sufficient precision, but that is not the case with the second. We are obliged to deduce it by means of an approximation to the velocity of some of the particles of the water of which the river is composed, and more particularly of the velocity observed in the surface, as well as the middle of the stream.

Amidst the numerous experiments made by Dubuat on the motion of fluids, we find some which have for their object to determine the proportion of the velocity on the surface with that at the bottom of the current, and its mean velocity. From these experiments he has deduced a formula with a sufficient degree of approximation; for putting V for the velocity at the surface, and U for the mean velocity, he finds this equation,

$$U = (V^{\frac{1}{2}} - \tfrac{1}{2}w)^2 + \tfrac{1}{4}w,$$

this last letter being constantly $= 0^{m}\cdot02707$; and to express the proportion between the velocity on the surface and that at the bottom, calling the latter W, he finds

$$W = (V^{\frac{1}{2}} - w^{\frac{1}{2}})^2.$$

These two formulæ are used by him to calculate the table that gives the estimate of the velocity at the bottom, and that of the mean velocity, proportionally to the velocity on the surface, from

$$V = 0^{m}\cdot027 \text{ to } V = 2^{m}\cdot707.$$

M. de Prony took up the subject after him,[1] and observed that the formulæ of Dubuat were in several cases at variance with the results of experience. For supposing in the second $V = 0$, we have $W = w$, from which it follows that the velocity at the surface being nothing, the velocity of the under stream would be equal to $0^{m}{\cdot}027$, a conclusion that cannot be admitted. If in the first formula we make $V = 0$, a finite value will be found for U, which also does not agree with the phenomena which Nature regularly presents. It necessarily follows that the relàtion established between U and V, must give at the same time $U = 0$ and $V = 0$, or $U = \infty$ and $V = \infty$.

M. de Prony having laboured to find out a formula in accordance with both these conditions, makes choice of an equation of this kind,

$$U = \frac{V(V + a)}{V + b};$$

and the values of the invariable quantities $a$ and $b$ being determined by seventeen experiments of Dubuat, made on the values of V, which varied from $V = 0^{m}{\cdot}15$ to $V = 1^{m}{\cdot}30$, by a method of correcting the anomalies explained in the work which we have quoted, he has found

$$a = 2{\cdot}37187,\ b = 3{\cdot}15312,$$

and the metre being unity in this measure, it gives for the preceding formula

[1] Recherches Physico-mathématiques sur la Théorie des Eaux Courantes, p. 73.

$$U = \frac{V\,(V + 2{\cdot}37187)}{V + 3{\cdot}15312}.$$

This formula has the advantage of representing with more fidelity, than that of Dubuat, the results of his own experiments, besides being in accordance with the phenomena in question, when considered in their proper bounds. M. de Prony remarks, that from $V = 0$, to $V = 3$ metres, the proportion $\frac{V + 2{\cdot}37187}{V + 3{\cdot}15312}$ is evidently equal to 0·82 ; and as practical cases are confined within these limits, the following formula will be found sufficiently exact :

$U = 0{\cdot}82.\,V$, or, which is nearly the same, $U = \frac{4}{5}\,V$.

This is the total amount of all that theory and experiment have hitherto discovered as the means of ascertaining the amount of the mean velocity of a course of water by means of the velocity of the surface and the current ; but something more is necessary to be observed respecting the manner in which the results we have explained may be applied to use.

The experiments of Dubuat have been made on artificial canals, the section of which was a rectangle or a trapezium, and in which the depth of the water varied from 54 millimetres to 27 centimetres. The results of these experiments must therefore be viewed with some reserve, and cannot be judiciously applied to the beds of rivers, as the nature of the motions of fluids is not sufficiently known to us as yet, so as to enable us to conclude on this point from small to great in a manner sufficiently accurate. Moreover, the experiments on

which the preceding formula is founded seem to indicate that the relations which exist between the three velocities, V, U, and W, are independent of the size and figure of the bed of the stream ; and these relations seemed not to undergo any sensible change, where the breadth of the bed was six or seven times greater than its depth.[1]

It is therefore difficult, as M. de Prony observes, to persuade ourselves that these different elements have no influence on the relative values of V, U, and W ; and experiments made on a larger scale, by throwing a new light on the subject, will compel us to have recourse to the conclusion already drawn from former observations.[2]

[1] Dubuat, Principes d'Hydraulique, tome i. p. 96.

[2] The author of these notes has published in the sixth volume of the Memoirs of the Academy of Sciences, the result of his inquiries on the motion of fluids, and has principally applied them to what is usually styled linear movement, that is to say, in the case where the fluid moves in a rectilinear bed, according to lines drawn parallel to the axis of the bed. It follows from these inquiries, with regard to the question which we now investigate, that whatever may be the figure of the transverse section, the mean velocity, and the greatest degree of swiftness which takes place at the surface and the middle of the stream, tend to an equality between themselves, in proportion as the dimensions of the bed become greater or less. Besides, the proportion between these two degrees of velocity is independent of their absolute value. If, in a rectangular bed, the horizontal breadth is supposed to be extremely wide, and the vertical depth of the water extremely small, the mean velocity will be about 0·64 of the greater. If both the dimensions of the rectangular bed be extremely large, the proportion becomes nearly 0·41. The formulæ present the

At the period of great floods, which it is necessary to choose, in order to arrive at the knowledge of the elements of calculation for the quantity of water which is discharged by the river, the banks are overflowed, and the water spreads on both sides to a great extent, and flows very slowly; while at the same time it moves with great rapidity in the middle of the current. It is highly probable that great errors would arise in such cases as these, from the adoption of the mode of calculation already laid down.

We must therefore select, if possible, a spot where the waters of the river are enclosed, or where, during floods, the overflowing is not considerable. The section of the stream's velocity at the passage of a bridge might also be safely taken, should it happen that there might be a bridge in the vicinity of the locality selected for the intended erection.

It is obvious that the methods which we have hitherto employed to obtain directly the amount of the mean velocity of a river, are defective, and liable to lead to errors more or less considerable. As a certain analogy

means of calculating the values of this proportion for various beds, the transverse sections of which might happen to be semicircles or rectangles.

Besides, experience proves that the actual laws relative to the motion of water in the beds of rivers, or in tubes, differ from those of linear motion; and the only case in which Nature realizes the latter laws, is where fluids move in rectilinear tubes of a very small diameter. The results immediately deduced from observation, are, in the present case, alone deserving of attention.

must exist between the section, the slope, and the velocity of a current, and as it is always possible to measure the slope and the section, the value of the velocity might be naturally ascertained, if this analogy were sufficiently known. In the work which we have already quoted, M. de Prony has deduced it from a variety of experiments, and has arrived at the equation

$$U = -0{\cdot}0719 + \sqrt{0{\cdot}005163 + 3232{\cdot}96\,RI},$$

in which U being always the mean velocity of the current, R will represent the mean radius, that is, the area of the section divided by that part of the periphery of this section which belongs to the solid in which the fluid flows; and I represents the slope or declivity per metre.[1]

[1] Subsequently to the experiments of M. de Prony, M. Eytelwein has published, in the Memoirs of the Academy of Berlin, the result of some new experiments, by which he has laid down a formula like the preceding, the coefficients of which have values dissimilar in some respects. This formula, taking the metre for linear unity, is

$$U = -0{\cdot}03319 + \sqrt{0{\cdot}0011016 + 2735{\cdot}66\,RI.}$$

The experiments which served to determine the coefficients comprehended values from 2 to $3^m$ for the velocity, while those values in the experiments made by M. de Prony did not exceed $0^m{\cdot}88$. The formula of M. de Prony agrees at least as well as that of M. Eytelwein with the experiment for the small velocities, but it appears rather defective with respect to the greater ones.

M. de Prony had given, for the motion of water in tubes, the expression

$$U = -0{\cdot}02488 + \sqrt{0{\cdot}0006192 + 2871{\cdot}43\,RI.}$$

This agrees better than that which is quoted in the text with the experi-

This equation will give the value of the mean velocity with sufficient exactness in every application, but it is necessary to remark that it proceeds essentially on the supposition that the extent of the section of the river, and the value of the declivity, are sensibly the same on a sufficient degree of length, so that the mean velocity therein may be regarded as uniform and regular; and it is requisite to pay attention to this consideration, whenever this formula is applied.

### *On the manner of fixing the outlet, or discharge, with reference to the mass of water in the river.*

The dimensions of the beds of rivers are generally uniform and similar. But with the exception of particular cases, in which they flow through sandy soils that yield with so much facility to the action of the water, that the current may, at every increase, stray into different places, there is a certain equilibrium established in every point of the course of the stream, between the action employed by the waters on the stone work, and the tenacity of the materials of which the bed is composed; and in virtue of this equilibrium no very remarkable changes take place either in the extent or the form of

ments in what the value of the velocity is above one metre, yet it leads to results of rather an equivocal nature. See the work entitled *Recueil de cinq tables pour faciliter et abréger les calculs relatifs au mouvement des eaux*, &c., published by M. de Prony in September, 1825. The memoirs of M. Eytelwein have been translated, and printed in the *Journal des Mines*, tome xii. 1826.

the bed, which constantly preserves the same ordinary nature. But if, from the operation of any cause whatever, the force with which the water tends to wear the bottom and the banks of its bed should be augmented, the bed will be forced to enlarge itself till the equilibrium shall be established afresh, unless the sides are composed of matter that presents a more considerable resistance than the force with which they are assailed. If, on the contrary, the velocity of the river should have experienced a diminution, its bed would be raised up by successive deposits, till the extent of the section would become such as to resume, with the resistance of the bottom, the proportions marked by nature for the steadiness of the river's course.

It results from these principles, that if the breadth of a river were diminished by works erected on its bed, its velocity would be increased, and it would form a slack water and a declivity, or shoot; the water would react on the bottom, and the bed would deepen till the enlargement of the section, combined with the increase of the velocity, would be capable of making an equivalent for the diminution of the breadth of the river, and till this velocity should find itself in equilibrium with the resistance at the bottom. If, on the contrary, an attempt should be made to increase the breadth of the bed, the velocity would be checked, and the bed would have a tendency to be choked up.

From these considerations, it is a most essential point, in the construction of a bridge, to pay attention to the

velocity which the waters will assume under the arches ; but this degree of velocity must not be augmented, to compel the current to react against the bottom of the river, and undermine the foundations of the piles and buttresses. Neither should it be sensibly diminished, because, in that case, the length of the bridge would be increased to no useful purpose, and this diminution would occasion deposits that might be dangerous in the end.

There are, nevertheless, some distinctions to be made with regard to the nature of the soil of which the bottom of the bed is formed. If it should be extremely compact and tenacious, and approach the nature of rock, it will not yield sensibly to the action of the water ; and whatever may then be its velocity, no apprehensions need be entertained that the bridge will break down. The only thing to be observed here, is, that the waters cannot assume a great velocity without producing in the upper stream a slack water, more or less considerable, and which, in cases where the outlet would be extremely confined, might produce inundations in the upper part of the river, and render its navigation difficult and dangerous. If the bottom is composed of matter that the water can easily master, care must be taken not to permit the velocity to be sensibly increased ; in short, if the soil yields to the action of the water with so much facility as to produce fears that during a flood the current would force its way under some of the arches, and penetrate the bottom, while it would deposit sand under

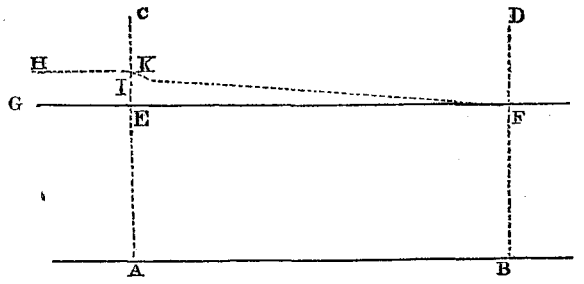
others, this would lead to the necessity of constructing a general frame of timber; then the bottom, offering in every direction the same resistance to the action of the current, would oblige it to assume a regular course, and distribute itself equally over the whole breadth of the river. As long as this frame should exist, no apprehensions need be entertained that any of the piles will be undermined.

It is evident, from the preceding remarks, that the velocity which the water may take under the bridge ought, in all cases, to be determined beforehand, either from the nature of the soil which composes the bottom of the river, or from the elevation of the slack water or eddies formed in the upper stream, on which the value of this velocity depends; and as the quantity of water impelled by the river is likewise known, the superficial extent of the outlet which the bridge shall have may be immediately established. The question is, therefore, reduced to the following problem: the section of the bed of a river, and the velocity of the stream, being given, to determine the mean velocity which the current shall acquire, and the elevation of the slack water or eddies, on the supposition that the bed is narrowed by the construction of the piers.

This problem is not susceptible of a rigorous solution, but we shall show, by neglecting some circumstances, the effects of which are not very sensible, and which compensate one another in a great measure, and by following the example of Dubuat, we shall obtain an

approximate solution which may be employed to great advantage in cases of this kind.

Suppose ACDB to represent the lateral face of a




pile, and GEF the natural inclination of the river before the construction of the bridge. As the current is confined within the interval EF, the velocity, and consequently the inclination of the river, will be in that part more considerable, and the surface of the water, omitting the particular resistances produced by the starlings, will take a direction that may be represented by the line IF. This surface, in the upper stream, will necessarily rise above the point I, and we designate it by the line HK, which is sensibly horizontal to a length of no great extent.

Let us call

$\Omega$, the natural section of the river;

$\omega$, the area of the section after the construction of the bridge, or the extent of the outlet;

V, the mean velocity of the water;

$v$, the mean velocity of the water under the arches, after the construction of the bridge;

I, the inclination of the river in metres;

$s$, the length of the piles and buttresses = AB or EF;

H, the elevation of the slack water = EK;

$g$, the accelerating force of gravity = $9^{m}\cdot809$.

We shall then have $v = \frac{\Omega}{\omega} V$, since, in any river, the velocities are in the inverse ratio of the area of the corresponding sections; and the heights, due to the velocities V and $v$, will be represented by

$$\frac{V^2}{2g} \text{ and } \frac{\Omega^2}{\omega^2}\frac{V^2}{2g}.$$

The part IK of the elevation of the slack water which corresponds with the augmentation of the velocity, will be

$$IK = \frac{V^2}{2g}\left(\frac{\Omega^2}{\omega^2} - 1\right);$$

and as, by reason of the contraction generally met with by the current at the passage of a bridge, the section $\omega$ is diminished, we replace in this expression the area $\omega$ by $m\omega$, designating by $m$ a coefficient, the value of which depends principally on the size of the piles, as well as on the form of the starlings and the formation of the arches. We shall then have

$$IK = \frac{V^2}{2g}\left(\frac{\Omega^2}{m^2\omega^2} - 1\right).$$

It will now be necessary to find the value of the inclination, which will be found by the length of the piles. Before the construction of the bridge the inclination was equal to $sI$, and as these inclinations increase nearly in

proportion to the corresponding velocities, we shall have for the value of the new inclination

$$sI\,\frac{\Omega^2}{m^2\omega^2}.$$

Thus the part of the elevation of the slack water which arises from the increase of the inclination under the arches of the bridge will be represented by

$$EI = sI\left(\frac{\Omega^2}{m^2\omega^2} - 1\right);$$

and we shall then have

$$EI + IK = H = \left(\frac{V^2}{2g} + sI\right)\left(\frac{\Omega^2}{m^2\omega^2} - 1\right)$$

for the elevation which the waters of the river assume in the upper stream, since their level must remain the same in the lower.

In order to apply the result which we have obtained, it will be sufficient to put in the place of V the expression $v\frac{m\omega}{\Omega}$, and to give afterwards to $v$ the value that is determined beforehand, after the principles laid down above for the velocity which the waters ought to take under the bridge.

With respect to the value of the coefficient $m$, we are far from being able to determine it with a desirable degree of exactness. The experiments made relative to this matter have been performed on orifices excavated in walls of different thickness, to which tubes were sometimes annexed, through which the water flowed away under a pressure of greater or less force. Besides, the

dimensions of these orifices, even according to experiments made on a large scale, are not to be compared with those of the arches of a bridge, and the circumstances of the vent or discharge are different from those that take place in the case that occupies our attention.

It will be afterwards seen, when we investigate the most advantageous form to be given to the starlings of the piles of bridges, that this form has a very sensible influence, as well as the thickness of the pile, on the contraction that takes place, and by which the natural flow of the river is modified when it comes in contact with the bridge. The value of the coefficient $m$, as we have observed before, depends principally on the form of the starlings, and the projections of the arches, when they are sunk under the water. We cannot assign with sufficient exactness that which it ought to assume in different cases, but we shall not be far from the truth in supposing $m = 0{\cdot}95$ when the piles terminate in a semicircle, or in acute angles; $m = 0{\cdot}90$ when they terminate in obtuse angles; $m = 0{\cdot}85$ when they terminate in right angles, supposing the arches to be wide. In the most disadvantageous cases, that is, in the case of small arches, and where the projections of the arches are sunk in the water, the value of the coefficient $m$ will perhaps be about $0{\cdot}7$. By putting in the preceding equation for H and $v$ their values found from the nature of the soil and other local circumstances, it will be easy to deduce that of the proportion of the two sections $\omega$ and $\Omega$.

In order to determine beforehand the elevation of the

slack water that may be formed in the stream above the bridge, it will be requisite to anticipate the changes that may arise in the elevation of the waters of the river, and the inundations resulting from it that may overflow the banks. According to the principles laid down by Dubuat, it is generally admitted that the surface of the water, which, previous to the construction of the bridge, and on the hypothesis of a uniform inclination, has a manifest tendency towards an inclined plane, becomes, after its construction, a concave surface, which comes into contact with the primitive surface where the rising of the water ceases. This author has even given rules for the calculation of the slack water, on the supposition that the section of this surface would form an arc of a circle of great magnitude. But on these investigations we cannot place implicit reliance. When the figure of the bed of a river is irregular, and presents considerable variations in the declivity of the bottom and its breadth, where, in the event of a flood, a part of the waters flow slowly, and on inundated banks, the exact determination of the figure assumed by the surface of the fluid cannot be obtained. But we can arrive at it with a sufficient approximation when the bed of the river is regular, and the motion of the water proceeds in a uniform manner. M. Bélanger, engineer of roads and bridges, has furnished a calculation in the following manner for this particular object. Let us call $s$ a length computed on the bottom of the bed, in the direction of the stream; $h$ the vertical height of the transverse section of the current; $\omega$ the area of this section, corresponding to the height $h$; $x$ the extent of the

section $\omega$, taken at the surface of the water; R the mean radius; $i$ the inclination of the bottom of the bed, which is supposed to be uniform; Q the volume of water discharged by the river in a second; and we have the equation

$$s = \int dh \, \frac{\frac{Q^2 x}{g\omega^3} - 1}{\frac{1}{R}\left(0{\cdot}00002427\,\frac{Q}{\omega} + 0{\cdot}0003655\,\frac{Q^2}{\omega^2}\right) - i},$$

in which the metre and the sexagesimal second are taken for unities of length and time, and in which $g$ represents the velocity $9^{m}{\cdot}809$, generated by gravity in one second. All the quantities comprised under the sign $\int$ may be estimated as a function of the elevation of $h$ in the section. In calculating the value of the integral indicated by this sign, setting out from the highest value of $h$, which takes place at the point of the rising of the stream, that is, immediately in the wake of the bridge, till we arrive at any fixed value whatsoever, which may be intermediate between the foregoing and the natural value of the section, the result will give the length comprised between the part of the stream above the bridge, and the section to which this second value of $h$ belongs. We may thus arrive at the knowledge of the relations between the given elevations of the sections, and the distances from the bridge where these elevations take place, and consequently, of the figure of the fluid's surface throughout the whole extent of the slack water. This extent is usually prolonged, to speak rigorously, to an infinite distance, but the rising of the stream ceases

to be evident at a limited distance.[1] Besides, the formation of slack water at the upper part of a bridge must occasion difficulties either with respect to the solidity of the structure, or the obstacles opposed to the navigation of the river; it ought therefore to be diminished by every possible means.

It has been observed above, that it is dangerous to give a river too wide an outlet, because in this case it might give rise to masses of alluvion, which, acquiring in time consistency enough to resist the action of the current, would, during floods, force the waters to pass under the arches that are less exposed to this obstacle, and occasion the piles to be undermined. For a similar reason, a bridge ought not to be composed of two parts separated by an island, as it is probable that one of the parts having been choked up, the whole current would be forced to flow to the other, which would lead to the destruction of the bridge. It was by an accident of this kind that the bridges of Chazey and Roanne were swept away. It may be additionally remarked, that bridges, in general, are never destroyed unless by some error in the outlet, and that ultimately too great a diminution of the section is always the cause of their ruin, either when a sufficient length has not been allowed to the bridge in the first instance, or, on the contrary, where a too considerable one has been given.

All the elements for calculating the area of the outlet

[1] It will be necessary, on this subject, to consult a publication by M. Bélanger in 1818, entitled "*An Essay on the numerical solution of certain problems relating to the permanent motion of running waters.*"

ought to be assumed at the period of great floods, and it is according to the quantity of water discharged at that time that this area should be determined. It must be observed, however, that it is expedient to arrange the arches (in rivers susceptible of this method) in such a manner that, during the lowest state of the current, there should be the depth of one metre under some of them, so that the navigation of the river shall not be impeded. It will be always possible to combine the different conditions of which the size and form of the arches may admit, and to arrive, in each particular case, at the best solution of the problem.

The preceding notions are easily applicable to the bridge now building over the Durance, at Bonpas. This bridge, which is of wood, is built between two mounds at a distance of 534 metres from each other. The breadth of the river, at low water, is only 110 metres, and its mean depth $1^{m}{\cdot}30$. But, during floods, the stream rises to 3 metres, and then the surface of the outlet is 1530 square metres.

As the bed of the river is not confined, and its breadth is considerable relatively to its depth, it was difficult to ascertain its mean velocity with any degree of exactness. But at the distance of about 4 myriametres above Bonpas, the Durance passes between two rocks, in shallow water, that are distant $180^{m}$ from each other. The elevation of the stream, at this spot, has been observed during considerable floods, and has been found to be :

1st, That the river having $2^{m}{\cdot}44$ of reduced depth, its mean velocity was $1^{m}{\cdot}95$ a second.

2nd, That when the river had $2^{m}\cdot92$ of reduced depth, the mean velocity was $2^{m}\cdot44$ a second.

3rd, That at the period of great floods, the depth of the water was $4^{m}\cdot87$, and its mean velocity $4^{m}\cdot12$. Thus, in the latter case, which is most worthy of observation, the discharge of the river is from $180 \times 4^{m}\cdot87 \times 4^{m}\cdot12 =$ 3612 cubic metres a second.

As the distance between Mirabeau and Bonpas is not considerable, and the river receives, between these two points, only some streamlets or torrents of no magnitude, there cannot be any great difference between the quantity of water that flows to both. But we may determine this difference by approximation, by comparing the surfaces of the basins that supply water to Mirabeau and Bonpas, where we shall find that the latter surpasses the other by $\frac{1}{16}$. Thus, at Bonpas, a volume of water passes in a second equal to 3838 cubic metres, which, in a section of 1530 square metres, gives a mean velocity of $2^{m}\cdot51$. Though the bridge is built of wood, it does not diminish, but in a slight degree, the superficies of the outlet; and it is observable that the velocity of $2^{m}\cdot51$ a second, which corresponds with an elevation of the water of more than $3^{m}$ in the bed of the river, scarcely surpasses $2^{m}\cdot44$, the velocity which the river naturally takes at Mirabeau for an elevation of $2^{m}\cdot92$. Thus the mean velocity at Bonpas is less considerable than in other parts of the river's course, and its waters move with great facility.

The outlet of the Rhône at the bridge of St. Esprit is nearly 3580 square metres, that is to say, a little more than double that of the Durance at Bonpas, while the super-

ficies of the basins that supply the Rhône at St. Esprit is five times greater than that of the basins that supply the Durance at Bonpas. Thus, though the velocity of the Rhône is very considerable under the arches of the bridge of St. Esprit, yet it appears that the velocity ascribed to the Durance at Bonpas is too great. But it will be observed that the latter river, at Bonpas, at no more than 25 myriametres from its source, is still a torrent, while the Rhône, at St. Esprit, is no longer so.

It is therefore a very essential point, in determining the vent or outlet of bridges, to distinguish rivers from torrents, and to bestow strict attention on the superficies of the basins, the nature of the soil, and the time which the water takes to pass. It may happen, as we have observed before, that it is necessary to construct a bridge of greater extent at a short distance from the river's source than at a place more remote from it, though it may have received many accessory streams in the interval.

## III. ON THE FORM OF THE ARCHES.

The arches of bridges are divided, with respect to their form, into three principal kinds, viz. : arches of a full semicircle, described by half of the circumference ; arches of a flat vault, usually described by several arcs of a circle of different radii, the form of which is nearly similar to a semi-ellipse ; and arches of a circular kind, which are formed from arches of greater or less magnitude.

Semicircular arches were anciently most in use. There are but few ancient bridges in which this form is not used, which for a long time has prevailed in Europe, and which has the advantage of presenting most solidity and facility in the construction. However, arches of this form are subject to the inconvenience of considerably obstructing the passage of the water.

The use of flat-vaulted arches was not introduced into France till the close of the seventeenth century. What led to the adoption of this form was the necessity of affording a wide discharge without considerably augmenting the height of the arches. It effectually answered this object, and when the two diameters are not very unequal, it presents as much solidity and facility in the construction as the full semicircular arch.

With respect to arches formed of circular arcs, it is necessary to make a distinction in the different cases of this form. The first is that in which the springings are under water, as is the case with most of the first great bridges built in France, such as the bridge of St. Esprit, and the ancient bridge of Avignon. In this case, the flat-vaulted arches have the disadvantage of giving a less considerable discharge, and admitting of very massy tympana. This latter defect seems to have been recognised by the first builders, for the backs of the arches are almost always filled simply with earth, or relieved by means of small arcades.

In the second case, the springings of the arc are raised perpendicularly nearly on a level with the highest water of the river, as may be seen in the bridge of Louis XVI.

at Paris, a circumstance which compels the architect to make the arc very low. Hence it happens that the lateral pressure of the voussoirs is very considerable, and it is necessary in such an instance to employ great care in the construction, so that the crown may not be subject to sink down after the arching, as sometimes happens. The manner in which this arch juts out is different from that which takes place in others, but such irregularities do not tend to overturn the piers, only giving them a horizontal declination. We shall hereafter point out the means which are most suitable for resisting this pressure without any extraordinary expense.

The resistance opposed by the piers to the current, when they are sunk, is one of the principal causes of the alluvions that are formed under the piles. Bridges built on the principle of the arc of a circle have, in this respect, a great advantage over the others, when the springings of the arches are not reached by the waters of the river.

It is hardly possible to lay down any general rules for the choice that is to be made of the different kinds of arches, as the decision, in each particular case, must depend on the various localities that may occur. The extent of the outlet which should be given to the river, the relative heights of the water at the greatest and lowest elevations, the height at which the surface of the pavement of the bridge can be placed, the obligation which is sometimes imposed, of allowing for the destruction of an arch, and consequently of making the

piles perform the office of abutments, are the principal circumstances that lead to a determination on this subject.

To the three kinds of arches of which we have spoken, and which are the only ones now in use in France, we must add the forms employed by the Arabians, and especially the Gothic form, composed of two arcs of a circle, and known by the name of ogee. The latter would have the inconvenience of diminishing considerably the outlet, a fault which is easily corrected by working apertures in the tympana, as has been done in one of the bridges of Pavia. There may be cases in which this form is to be preferred, but taste ought to proscribe none, as all of them possess merit when suitably applied.

## IV. OF THE SIZE OF THE ARCHES.

Although the size of the arches usually depends on the particular circumstances of the locality of the intended bridge, yet we shall venture to sketch out a few ideas that may elucidate this point.

Small arches are best suited to quiet rivers, the waters of which do not rise to any considerable height. It is, in that case, easy to fix the foundations, and it furnishes an additional reason for not fearing to multiply the points of support. Large arches, however, are most suitable to torrents, in which it is generally difficult to lay the foundations, and in which the waters frequently draw down rocks and trees, which may damage the piles and the

abutments of the arches, which, in that case, ought to be placed above the level of the water.

In large rivers, wide arches ought in general to be adopted in preference, especially when they are subject to inundations, or sudden risings of the stream; but the method, whether more or less expensive, used in forming or establishing the foundations, will influence this determination. Regard must be also paid to the nature of the materials to be employed in the construction of the bridge, which ought to afford more solidity for large arches than small ones, and also to the nature and size of the vessels that navigate the river, which will require a free and convenient passage.

With respect to the width to be allowed to arches, there are two plans to be pursued: the apertures may either be all equal, or they may progressively diminish from the middle arches to those that join the abutments.

If all the arches are equal, there is the advantage of giving to the tops of the vaults the same elevation above the water, and of using the same centering which served for the two first. But then it is necessary to raise the approaches to the bridge, and it may consequently become requisite to make considerable embankments, and possibly to endanger the houses that may be in the vicinity. The rain water that is deposited for a length of time on the bridge is not easily got rid of, as it penetrates down to the coat, and undermines it.

When the diameters of the arches are unequal, these latter inconveniences disappear, and we are then at liberty to give to each side of the pavement an inclination

which ought not to exceed three centimetres per metre.[1] We thus diminish the obstacles to the approaches, and the height of the embankments. It is, moreover, possible to unite the advantages of both these methods, by giving the same width to all the arches, and placing the springings at heights decreasing from the middle to the extremities of the bridge.

It is necessary to give a sufficient height to the arches, so that in floods, extraneous matter, such as trees, carried down by the stream, may pass freely under them. The minimum of this height, when the arches are equal, is nearly one metre; when they are unequal, the height lies between 70 centimetres and $1^{m}{\cdot}4$.

## V. OF THE BREADTH OF BRIDGES.

The breadth that ought to be allowed to bridges depends wholly on the locality in which they are situated. It ought to be regulated by the degree of importance attached to the road on which they are built, or the population of the town which they benefit, and it is an essential point not to make this breadth too considerable, as it leads to a superfluous expense.

If the bridge is built in the country, and for a neighbouring road, it will be sufficient to allow it from 4 to $5^{m}$, especially if it is not very long. For a road of the second class, the breadth ought to be from 6 to $7^{m}$,

[1] This is 3 in 100. See Table of English and French Measures, p. 39.

which will allow room at once for two carriages, and for foot passengers, to pass. An extent of from 9 to 10 metres ought to be allowed to a bridge on a road of the first class.

In the interior of cities, the breadth of the bridge may vary from 10 to 20 metres, with respect to the population, and the activity of commercial intercourse, but it ought not to exceed the latter limit. The Pont-Neuf, at Paris, is undoubtedly one of the greatest thoroughfares in the world, and its road-way, which has a breadth of 22 metres between the parapets, is by no means too confined.

## NOTE

### *On the degree of velocity necessary to currents to support and convey different matters.*

We shall quote beforehand some estimates of the velocity of currents, observed under different circumstances, which may furnish some terms of comparison.

| | |
|---|---|
| The ordinary velocity, per second, of the small rivers in the vicinity of Paris, the declivity being 0·00018 . . . . . | 0·28 per metre. |
| The velocity of the Seine between Surène and Neuilly, observed by M. de Chézy, the elevation over the low water being $1^{m}$·26, and the declivity 0·000125 | 0·78 |
| The velocity of the Seine, in the interior of Paris, the water being at $0^{m}$·6 at the bank, and the declivity 0·00055 . . . . . | 1·00 |
| The same, the water being $6^{m}$ at the bank, and the declivity 0·0006 . . . . . | 1·90 |
| The greatest velocity of the Thames, at London, during high water . . . . . | 0·90 |
| At low water . . . . . | 0·76 |
| The velocity of the Tiber, at Rome, at low water . . | 1·00 |
| The velocity of the Danube, at Ebersdoff, at low water . . . . . | 1·05 |
| At high water, this velocity varies from $2^{m}$·21 to $3^{m}$·79. | |
| The velocity of the Loire, the declivity being 0·000382 | 1·30 |
| The velocity of the Rhône, at Arles, at low water . | 1·46 |
| The velocity of the Rhône, at Beaucaire, the same time . . . . . | 2·60 |
| The ordinary velocity of the Durance, from Sisteron to its mouth, the height of the water at the bank not exceeding $3^{m}$ . . . . . | 2·60 |

The velocity of the Maragnon at the pass of the Pongo, observed by M. de la Condamine . . . 3·90 per metre.

The velocity of a torrent arising from the melting of snow caused by the eruption of a volcano, observed in America by Bouguer . . . . . 7·80

We now insert a table, furnished under the article "Bridge," in the Edinburgh Encyclopedia, by Messrs. Telford and Nimmo, indicating the velocity of different currents, and the nature of the materials likely to yield to their action.

| The ordinary nature of currents. | Velocity per second. | Materials that resist these velocities, and yield to more powerful ones. |
|---|---|---|
| | Metres. | |
| Very slow . . . . . | 0·076 | Wet ground, mud. |
| Gliding . . . . . | 0·152 | Soft clay. |
| Gentle . . . . . | 0·305 | Sand. |
| Regular . . . . . | 0·609 | Gravel. |
| Ordinary velocity . . . . | 0·914 | Stony. |
| Extraordinary and rapid floods | 1·022 | Broken stones, flints. |
| Extraordinary floods, and rapids . . . . . | 1·052 | Collected pebbles, soft schistes. |
| | 1·083 | Beds of rocks. |
| Torrents and cataracts . . . | 3·005 | Hardened rocks. |

We shall next insert the results of the experiments made by Dubuat, in order to discover the regular velocity that best suits soils of different natures, the materials of which are placed at the bottom of an artificial canal formed of thick planks.

| Velocity per second. | Materials that resist these velocities, and yield to more powerful ones. | Specific gravity. |
|---|---|---|
| Metres. | | |
| 0·081 | Brown clay, fit for pottery . . . . . | 2·64 |
| 0·108 | Large gravel, like a grain of aniseed . . . . . | 2·54 |
| 0·162 | (The preceding materials are carried away.) | |
| 0·189 | Gravel as large as a pea, at most . . . . . | the same. |
| 0·217 | Large yellow sand . . . . . | 2·36 |
| 0·325 | Gravel as large as a Marsh-bean . . . . . | 2·54 |
| 0·474 | (The preceding materials are carried away.) | |
| 0·650 | Round pebbles of the sea-shore, of $0^{m}{\cdot}027$ diameter, at most | 2·61 |
| 0·975 | Angular flints, as large as a pullet's egg . . . . . | 2·25 |
| 1·220 | (The preceding materials are carried away.) | |

From these experiments it may be inferred, that the materials most commonly found in the beds of rivers, such as sand and gravel, are carried away by very low velocities. It follows that these materials are, in general, in a constant state of motion. They flow away, as well as the water, but more slowly, and by a species of particular change of place, which has been well observed and described by Dubuat. But it is not necessary, in order that a bridge may not be undermined, that the velocity of the water under the vaults shall not surpass the velocity of the current that may carry away the materials which form the bottom of its bed; but it is necessary that this velocity shall not sensibly surpass that which takes place above and below the bridge.

ENGLISH AND FRENCH MEASURES.

| ENGLISH. | FRENCH. |
|---|---|
| Inch (36th of a yard) | 2·539954 centimètres. |
| Foot ($\frac{1}{3}$ of a yard) | 3·0479449 décimètres. |
| Imperial yard | 0·91438348 mètres. |
| Fathom (2 yards) | 1·82876696 mètres. |
| Pole or perch ($5\frac{1}{2}$ yards) | 5·02911 mètres. |
| Furlong (220 yards) | 201·16437 mètres. |
| Mile (1760 yards) | 1609·3149 mètres. |
| **FRENCH.** | **ENGLISH.** |
| Millimètre | 0·03937 inches. |
| Centimètre | 0·393708 inches. |
| Décimètre | 3·937079 inches. |
| Mètre | 39·37079 inches.<br>3·2808992 feet.<br>1·093633 yards. |
| Kilomètre | 1093·63 yards. |
| Myriamètre | 6·2138 miles. |

| SYSTEMATIC NAMES. | VALUE. |
|---|---|
| ITINERARY MEASURES. | |
| Myriamètre | 10,000 mètres. |
| Kilomètre | 1000 mètres. |
| Décimètre | 10 mètres. |
| Mètre | The basis of the weights and measures. Ten millionth of the quadrant of the terrestrial meridian. |
| LONG MEASURE. | |
| Décimètre | 10th of a mètre. |
| Centimètre | 100th of a mètre. |
| Millimètre | 1000th of a mètre. |

## ON THE DESCRIPTION OF THE ARCHES OF BRIDGES.

### VI. ARCHES OF A FULL SEMICIRCLE.

We have already explained the principal advantages and inconveniences of arches of this kind. No further remark is necessary with respect to their description, as their form is entirely determined when the aperture is settled, since this form is a semicircle of which the aperture is the diameter. It is sufficient to observe that the centre of the semicircle is usually situated at the elevation of the foundations, or that of the low water. When circumstances admit of it, it may be placed below this elevation, and then the arch is perpendicularly erected.

### VII. ARCHES WITH FLAT VAULTS.

This form of an arch is not entirely determined when the span, and even the elevation, have been already fixed. It is, in fact, possible to describe on two given diameters an infinite number of different curves.

The only conditions to which the curve of flat-vaulted arches is subjected, are, that the tangent at the vertex should be horizontal, and that the tangents at the springings should be vertical. As the semi-ellipse satisfies both these conditions, it seems natural to choose this curve, especially as the curvature decreases regularly from the springings to the vertex, and forms an agreeable object to the eye. But it has the inconvenience in the construction of obliging the architect to change the form of the voussoirs that compose the arch, a circumstance which is very unpleasant, and has, besides, the defect of not allowing so wide an outlet as the curves of which we are about to treat, unless the difference between both the diameters of the arch be not very considerable.

Instead of the semi-ellipse, curves composed of a certain number of arcs of circles are usually employed, because the builder is then at liberty to determine conveniently the length and the radius of these arcs, and to give to the flat-vaulted arches the form which may be found most suitable and convenient.

In this case, both of the following conditions are necessary to be observed, viz.; 1st, that the line of the first arc, setting out from the abutments, shall include that of the ellipsis on the two diameters of the flat-vaulted arch, so as to afford to the arch a wider scope of outlet than it would have without this ellipsis; 2nd, that the radius of the vertical arc shall not exceed a certain limit. But the value of this limit cannot generally be determined, though it should scarcely

exceed the span of the arch; and if it be found necessary, from particular circumstances, to employ a greater radius, care must be taken not to direct the joints of the arch-stones to the centre of the arc, but towards a nearer point.

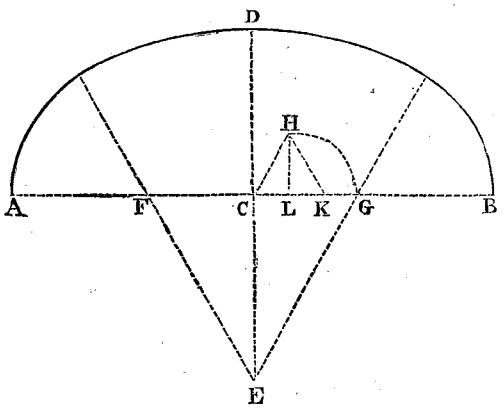
After settling these two first conditions, the next step will be to fix the number of arcs that are to enter into the composition of the flat-vaulted arch; they must not be less than three, and it has never happened that more than eleven have been used.

We shall now explain the mode of description in all these different cases.

*Flat-vaulted arches described with three arcs of a circle.*

The length of the radii of the three arcs that are to compose the flat-vaulted arch not having been fixed when we determined the two diameters of the curve, another condition must be employed. But it will be first necessary to suppose that the three arcs make each sixty degrees, that is to say, each of them equal to the sixth part of the circumference.

Let the half of the aperture of the arch AC=$a$, and its elevation CD=$b$. The centres of the two arcs described from the springing of the arch-stones will be then situ-



ated in the points F, G, in the line AB, and the centre of the third arc in the point E of the line DC produced. One of the centres being found, the two others will be easily determined. Let us call $x$ the distance of the point C, the middle of AB from the centre G, then as the triangle FGE is equilateral, EG=FG, and consequently,

$$CE+CD=FG+GB,$$

in which

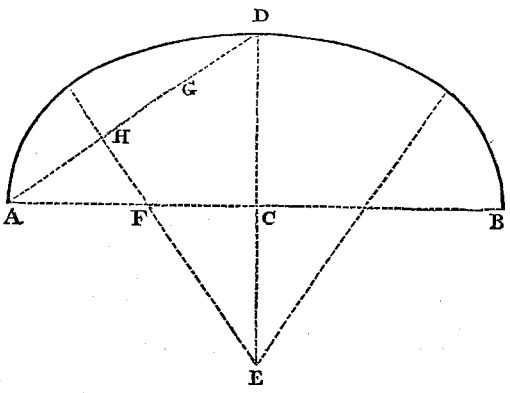
$$\sqrt{3}.\ x+b=x+a,$$

and resolving this equation,

$$x=\frac{a-b}{\sqrt{3}-1}=\frac{\sqrt{3}+1}{2}(a-b).$$

To construct this equation take $CK=a-b$, and upon it describe the equilateral triangle CHK, let fall the perpendicular HL, and from L apply LG=LH the altitude of the triangle, and G is the required centre.

Let us now suppose a condition to be laid down that the radii of the great and the small arc are different in the least possible degree from each other. Let $AC=a$, $CD=b$, the radius of the arc at the abutment $y$, and $x$ that of the vertical arc, the centre of which is in E, and we shall have, by means of the right-angled triangle CFE,




$$(x-y)^2=(x-b)^2+(a-y)^2.$$

By solving this equation for $x$, we find

$$x = \frac{\frac{1}{2}(b^2+a^2)-ay}{b-y};$$

we shall then have for the expression of the proportion of the two radii,

$$\frac{x}{y} = \frac{\frac{1}{2}(b^2+a^2)-ay}{by-y^2};$$

by making its differential equal to zero, and finding the value of $y$, we have, after putting, for the sake of conciseness, $b^2+a^2=c^2$,

$$y = \frac{bc}{c+(a-b)};$$

this substituted in the above, gives

$$x = \frac{ac}{c-(a-b)}.$$

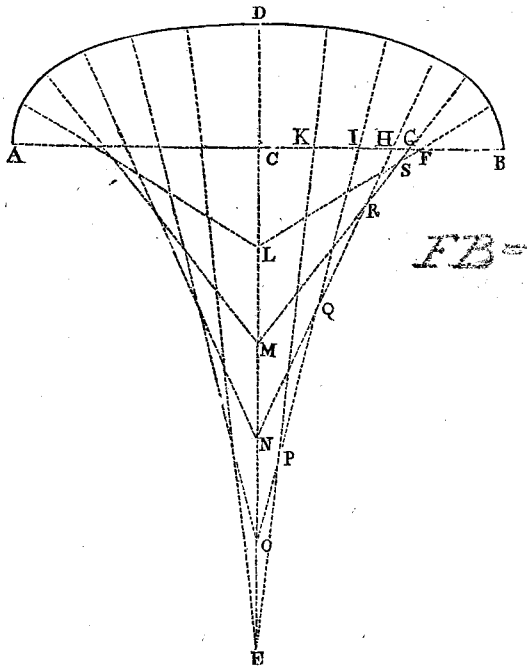
These two expressions are constructed by taking from the line AD the distance DG equal to $a-b$, and raising on the middle of the remainder AG, the perpendicular HE; then the points E, F, at the intersection of the two diameters will be the centres sought. This method gives a greater difference between the extent of the two arcs than the foregoing one, which appears preferable.

When the proportion between the two diameters is not less than one third, the difference of the radii of the arcs of which the flat-vaulted arch is formed is not great enough to make the transition from the one to the other too visible, and to produce a disagreeable effect; in that case it will be sufficient to describe it with three arcs of a circle. But when the arch is lower, then it will be necessary to employ a greater number.

*Of flat-vaulted arches described with more than three arcs of a circle.*

Different methods have been adopted in order to determine the positions of the centre, and the length of the radii of the arcs of which these curves are to be composed. We shall explain the mode which was preferred for the plan of the arches of the bridge of Neuilly.

After having determined the radius FB of the first arc, setting out from the abutments, a distance was taken for the prolongation of the less diameter; call this distance CE, which is arbitrarily made equal to three times CF, and which may have a different proportion to this line. Afterwards dividing CE into five equal parts, CF into five parts in proportion with the numbers 1, 2, 3, 4, and 5, and joining the points of division by the lines LF, MG, NH, OI, EK, the points E, P, Q, R, S, F, which are found at the respective intersections of these lines, were assumed for the centres of the different arcs that compose the flat-vaulted arch.




It is evident that in the curve described in this man-

ner the proportion of the elevation CD at the aperture A B, depends on the foregoing data, assumed from the length of the line CF, and its proportion with respect to CE. But, when a flat-vaulted arch is to be described, the two diameters are usually determined beforehand, so that, after describing a curve by the preceding method, it will be requisite to modify it in such a manner that the elevation CD shall become precisely equal to the one that is assumed.

Let us call half of the aperture in question $a$, and its elevation $b$, $x$ the extent which CF is to have, and $y$ that which is to be assigned to CE. Let us suppose, besides, that the assumed and arbitrary values given to CF and CE be represented by $n$ and $m$, and that these results form, for the developement of the portion of the polygon EPQRSF, a length equal to $s$, while the length really to be described on the lines $x$ and $y$ will be equal to $z$.

We then have the relation

$$z+a-x=y+b\ ;$$

and if we suppose the figure constructed on the lines $x$ and $y$, to be similar to the figure ECF constructed on the lines CF$=n$ and CE$=m$, we shall have

$$y=\frac{mx}{n},\ z=\frac{sx}{n}\ ;$$

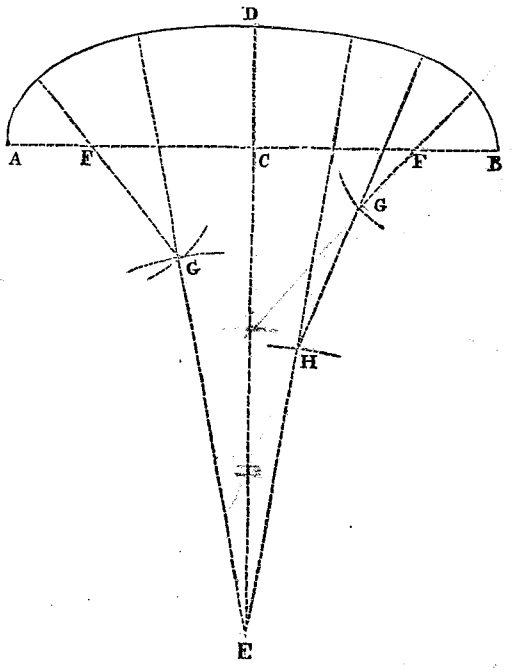
substituting these values in the foregoing equation, and deducing that of $x$, we find

$$x=\frac{n\,(a-b)}{m+n-s}\ :$$

this is the value that must be given to CF, in order that the aperture and the elevation of the flat-vaulted arch shall be precisely equal to the lines represented by *a* and *b*.

It is plain that this method may be applied to flat-vaulted arches, composed of any number of arcs whatever. By means of the same centres parallel curves may be described, in which the proportion of the diameters will be variable, and if the proportion between the distances CF and CE be suitably settled, then curves will be found which, described for the same arcs, will furnish forms of great variety, and allow a more or less free passage to the water.

The foregoing method has no inconvenience but its length, and leaves little to be desired; but it is imagined that in the case in which the arch should be lowered one-fourth it is fruitless to compose the flat-vaulted arch with a great number of arcs of a circle, and that in general it will be sufficient to confine their number to five; in this case the description is simplified in a great degree.




Let us suppose that the lengths AF and DE, the radii *r* and R of the arc at the buttresses, and the vertical,

are previously determined. Let us call the radius of the intermediate arc $\rho$; we can determine it by the condition to be a mean proportional between $r$ and R; we shall then have $\rho = \sqrt{\mathrm{R}\,r}$. Then describing from the point F as a centre, and with a radius equal to $\rho - r$, an arc, and from the point E, and with a radius equal to $\mathrm{R} - \rho$, a second arc intersecting the former in G, the point G shall be the centre of the arc which will reunite that at the buttresses with the vertical one.

If it should be necessary to describe the flat-vaulted arch with seven centres, it will be proper to find two mean proportionals $\rho$ and $\rho'$ between the radii of the extreme arcs $r$ and R, which will give

$$\rho = \sqrt[3]{\mathrm{R}r^2} \text{ and } \rho' = \sqrt[3]{\mathrm{R}^2 r}.$$

Then describe from the point F as a centre, and with a radius equal to $\rho - r$, an arc which will contain the centre of the arc of which $\rho$ is the radius; and from the point E, with the radius $\mathrm{EH} = \mathrm{R} - \rho'$, a second arc which will contain the centre of the arc of which $\rho$ is the radius. In order to fix afterwards on each of these arcs the respective positions of the two centres, it will be necessary to draw between both these arcs a line HG, the length of which shall be equal to $\rho' - \rho$; but, as the position of this line is not determined by this condition only, it must be fixed by means of a tact on the part of the architect that may be guided by the condition, that the extent of the arcs of which the flat-vaulted arch is to be composed shall decrease gradually and uniformly from the vertex to the springing.

This method may be extended to flat-vaulted arches composed of a greater number of arcs; but it is always nearly useless to employ more than five.

When an extensive combination of arcs is to be formed, it is not possible, unless in the parts near the springings or abutments, to employ the beam-compasses to strike the necessary arcs. The mode is, to begin by fixing the extremities of each of these arcs, the co-ordinates of which have been calculated beforehand, in such a manner as to form an angle, the supplement of which shall be equal to the half of the arc. This angle must be moved in such a way that its sides shall always pass through the extreme points of the arc, the vertex of which gives the intermediate points.[1]

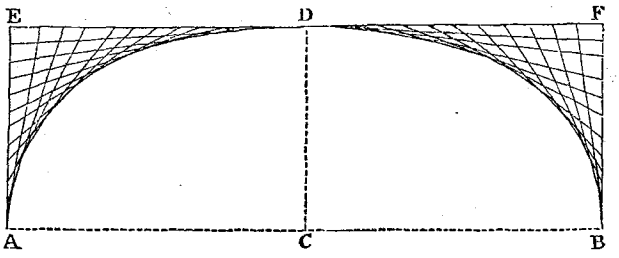
### *Of flat-vaulted arches which are not described with arcs of a circle.*

The difficulty of tracing out on a large scale, and in a manner perfectly exact, the curve which is projected, when it is composed of several arcs of a circle, has given rise to different methods of describing the flat-vaulted arch, in which this difficulty almost totally vanishes.

Carpenters usually employ, in order to adapt a curve

[1] For a full description of this as well as different other methods of describing arcs of a circle, see Dr. Gregory's Mathematics for Practical Men, pp. 127, 128.

to two sides of an angle AED, a method which consists in dividing the two




sides of the angle into a number of equal parts, and to join the points of division by lines which they consider as tangents to the curve, and which, being supposed to approximate infinitely, determine each of their points by their successive intersections. By performing this same operation on the angle BFD, a portion of a curve will be found equal to the first, and which will complete the description of the arch ADB.

It has been observed for a length of time that the curve described in the foregoing manner was a portion of a parabola, the vertex of which is situated between the points A and D; and it allows of a wider outlet than is afforded by a flat-vaulted arch composed of three arcs of a circle, or by a full semicircle, which should be described on the same axes. It thus presents an advantage in this respect as well as in the facility of description. It retains however an inconvenience resulting from the more or less agreeable aspect which curves of this kind present; in fact, it is evident that in the parabola the value of the radius of curvature is a *minimum* at the vertex of the curve; and by reason of the plane where the vertex is fixed in this instance it follows that the curve does not diminish progressively from the springings to the most elevated point of the

arch. It has besides been proposed to form the flat-vaulted arch from two arcs of a circle described from the line of the springings, and connected at the vertex by a portion of the catenary curve. Curves composed in this manner present, particularly in the lower part, a much wider outlet than the ordinary flat-vaulted arches, and in this respect they seem to be deserving of a preference.

## VIII.—ARCHES FORMED FROM AN ARC OF A CIRCLE.

Latterly a very considerable number of bridges has been constructed in which the sweep is described by an arc of a circle, but care has been always taken to place the springings nearly on a level with high water. Their positions being fixed by this condition, if the height to which it is possible to raise the vertex of the arch is equally afforded by the localities, the arc is then completely determined. But very possibly, in this case, the opening would be too low, or the radius of the arc may have too great a length, so that the work would not present that solidity which is requisite to be obtained in such a construction as a bridge. In that case, it will be best to abstain from the use of the arc of a circle, and from the advantages which, in other points of view, it might afford.

## IX. — ON THE SUBSTANCE WHICH SHOULD BE GIVEN TO THE ARCH-STONES OF BRIDGES.

After having determined the curvature of the arches, the next question which presents itself is that of the depth of the stones of the arches at their crown; because it is upon this that the extent and the direction of the thrust depend, and consequently the resistance of which the piers and abutments should be capable. In regard to this point the works of the ancients, and even those of the moderns, exhibit very great differences. Several engineers have, indeed, endeavoured to establish precise rules for this purpose, yet as such rules are not founded upon true and obvious principles, they have not been strictly adhered to. We shall now proceed to state some particulars briefly, on those which are most generally known.

The principal Italian architects, such as Alberti, Palladio, and Serlio, have vaguely indicated a few of them: one having fixed a twelfth, one a fifteenth, and another a seventeenth, of the span or chord of the arch, as the proper proportion for the depth of the stones at the apex. It is obvious enough, however, that although they are not false in themselves, because founded upon actual examples from executed structures, such general indications are not at all to be relied upon whenever the ordinary forms and dimensions are departed from; for they are given by their authors without their entering

into their reasons for doing so, or supporting them by arguments of any kind.

It is the same with regard to the rules given by Gauthier in his Treatise on Bridges; who distinguishes vaults accordingly as they are constructed either of hard or of soft stone; and to the former he allows a fifteenth of their span, when it exceeds 10 metres (32·8099 English feet); and to the others about 32 centimetres (12·5986 inches) more. Yet he takes no account of the different forms of curvature; and how defective his rule is, may be seen at once, when we find that the arches of the Pont de Neuilly support themselves with a depth of no more than 1·62 metres (5·315 feet) at their crown, whereas, according to Gauthier, it ought to be 2·6 metres (8·53 feet).

Boffrand has also drawn up tables for the same purpose, which give in general a greater ratio for the depth of the stones at the crown of an arch than Gauthier does, and consequently are no more deserving of attention.

In Perronet's work we meet with a rule which assigns for the depth of the arch-stones at the crown of an arch one twenty-fourth of its span; to which 325 millimetres (12·795 inches) are added, afterwards subtracting $\frac{1}{144}$ of the chord or span. This rule will be found to agree generally with the depths adopted in such bridges as have been built, especially in those whose arches are semicircular or segments of circles; yet some arches which have been executed appear to prove that the rule in question establishes too great a substance for the

structure, when the opening or span of the arch exceeds 30 metres (98·427 English feet).

We[1] are therefore of opinion that for a rule of this kind the following ought to be preferred: 1st, a depth of 0.33 metre (12·9923 inches) to all arches of less than two metres (6·5616 feet); 2ndly, for those from two metres (6·5616 feet) to sixteen metres (52·4928 feet) the depth should be $\frac{1}{48}$ of the span, increased by 0·33 metre (12·9923 inches); 3rdly, for those from 16 metres (52·4928 feet) to 32 metres (104·9856 feet), it ought to be $\frac{1}{24}$ of the span; 4thly, for arches exceeding 32 metres (104·9856 feet) the depth should be $\frac{1}{24}$ of the first 32 metres, and $\frac{1}{48}$ of whatever may be the overplus. The same rule will also serve both for semicircular and segmental arches, and to those of flatter and compounded arcs. There are many existing bridges of great size, the substance of whose arches is less than would be deduced from the foregoing rule.

In order duly to determine the substance of an arch, regard should be had both to the materials and to the mode of construction employed.

When it shall have been seen what are the effects which manifest themselves in arches after they are constructed, we shall be convinced that if an arch could be formed of incompressible materials, it could not settle, except in such degree as the resisting parts might not be sufficiently strong to support those which press upon

[1] '*We*,' in these Papers, must not be understood to mean the Editor, Editors, Author, or Authors of the work generally.

them; or, in other words, only in such degree as the abutments might not be sufficiently firm to resist the thrust of the arch.

Free-stone (*pierre de taille*) may be considered, for all practical purposes, as incompressible; and we find that if the arch-stones were to be placed one upon another without mortar, and without there being any other filling in or packing in such manner that there could be no bulging at all, the arch would support itself, provided the abutments were sufficiently firm, and the key-stone of such substance that it could not be crushed or splintered by the pressure it would have to sustain.

The depth of the key-stone would then at once be determined, if we knew the horizontal pressure the two halves of the arch exert upon each other, and the power of resistance of the stone they are constructed of; but it would be necessary to take into account the shaking from the motion of carriages to which such arches are exposed, in order not to give them an inadequate substance, more especially those of small diameter, upon which the pressure is less considerable.

If the pressure borne by the arch-stones in the crown of an arch of the boldest execution be calculated, we find that it is much less than what would be requisite to crush the stone. In the Pont de Neuilly, for instance, the horizontal pressure against the key-stone is about 14,000 kilogrammes (30,876·72 lbs. av.) for each metre of its depth, supposing the masonry cut down to the level of the summit of the extrados; or about 185,000

kilogrammes (408,013·8 lbs. av.), having reference to the weight of the filling in and of the pavement, and to that of carriages and passengers which it may have to sustain occasionally. The arches are built of Saillancourt stone ; and according to M. Rondelet's experiments, a cubic piece of this stone of the first quality, whose side is 5 centimetres (1·9685 inches), requires a pressure of 3500 kilogrammes (7719·18 lbs. av.) to crush it. Consequently, as the arch-stones at the summits of the arches are 1·624 metre (5·328 feet) deep, the force necessary to crush the stones, supposing it proportioned to the surfaces, would be 2,275,000 kilogrammes (5017467 lbs. av.), that is, a pressure more than twelve times greater than what it is exposed to.

In the arch proposed to be built at Melun, with a span of 48·7 metres (157·4832 English feet), formed by the arc of a circle whose radius would be 65 metres (213·25 feet), the horizontal pressure against the key-stone would have been about 240,000 kilogrammes (529,315·2 lbs. av.) for every metre of its depth, on account of the specific gravity of the stone (hard conglomerate sand-stone, *grès dur*), and of occasional additional weights. But if we take only 15,000 kilogrammes (33,082·2 lbs. av.) as the resistance of this kind of stone on a surface of 25 square centimetres (3·875 square inches), which is below the results published by Rondelet, the force requisite to crush a stone 1·624 metre (5·328 feet) deep, would be 9,744,000 kilogrammes (21,490,197·12 lbs. av.), which is equal to more than forty times the actual pressure.

After this it would appear that we might adopt for

arches a considerably less degree of depth in the arch-stones than what our most experienced constructors usually give them. But these calculations suppose that the arch-stones press equally on each other throughout, and that after the construction is completed there is no play or vibration, and no greater pressure on one part than on another. This hypothesis would not be very far from the truth if the joints of the arch-stones were perfectly executed; and if packing were avoided, adopting indeed the method employed by the ancients in their works of the same kind, we may conclude that bridges might be constructed possessing greater boldness and lightness than are possessed by the generality of those which have hitherto been executed by the moderns.

Nevertheless, both the change produced in the stones themselves by the effects of time, and the inevitable imperfection in the execution of the joints, would alone deter us from reducing the key-stones of bridges to the substances, which, according to such calculation, would appear to be sufficient, because the slightest splintering or fraying would sensibly diminish their resistance, and possibly endanger the arch. But the practice of packing the joints, and the liability to settling which results from it, and hence again of changes in the forms of arches, appear to be the principal causes of the great differences between the substance indicated as necessary by the strength of the stone, and that which is usually given.

In effect, the packing being, previously to the mortar becoming perfectly dry, the only intermediate substance

by which the arch-stones bear one upon another, it is evident that the resisting surface is not that of the entire surface or bed of the joint, but that it is reduced to that of the packing. Hence, it would seem possible to determine the depth of the arch-stone, in such manner that the superficies of the packing on which it rests were sufficiently large that the corresponding parts of the joint of the stone would be able to resist the pressure. But, it is necessary to remark, that the power of resistance of the stone cannot in this case be computed from experiments already known, as these have been made on small detached cubes.

It may also be remarked that the vertical settling of an arch is, *cæteris paribus*, greater in proportion as the depth of the key-stone is diminished. This settling depends, in part, on the shortening of the curve DE*d*, which connects the supporting points or extremities of the arch-stones, or those about which the stones have a tendency to turn in the settlement of the arch. This curve passes through the points of rupture D and *d*, and through E the summit of the extrados, and it is more or less depressed according as the key-stone is more or less deep. Hence we may easily perceive that, if the compression of the packing were known and the shortening the curve undergoes thereby, with regard to a determined pressure, we could compute the amount of settlement for an arch, corresponding to the given depth of its key-stone. The settlement would be greater in proportion as the curve DE*d* might be depressed: but the more considerable the settlement may

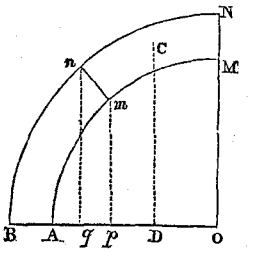
be, the more it may be found that it will be irregular, and that the form of the arch will become changed by means of it. This consideration induces us to endeavour to diminish the liability an arch may be under in this respect, by giving the key-stone more than depth enough to yield sufficient resistance to the greatest amount of pressure.

After what has been said it does not appear that it would be possible to establish any rule for determining the depth of the stones for the arches of bridges; besides, this question does not seem to admit of any general solution, because it evidently depends upon the nature of the materials, and upon the mode of construction employed. If the arch-stones were to be wrought according to the manner practised by the ancients, the beds being rubbed to an even surface, it would be a matter less arbitrary and uncertain. We might then reasonably presume that the arches would not settle at all when once constructed; and it would be sufficient to calculate the power of resistance of the stone, and its greater or lesser power of resisting the effects of time, and accidental injuries. But the mode usually practised does not admit of such calculation being applied, in the absence of experience, to all the elements of which it is composed. We must, therefore, content ourselves with consulting the examples with which the most skilful constructors have furnished us.

## X.—ON THE PRESSURE TO WHICH THE STONE-WORK OF ARCHES IS EXPOSED.

It will be readily acknowledged that the question as to the degree of depth that ought to be allowed to the arch-stones of bridges is combined with a great variety of considerations. It is clear, however, that the greatness of the pressures to which key-stones are exposed, and the degree of resistance presented by the stone-work, form the principal elements of this inquiry, and that it is extremely necessary to pay attention to these points in the formation of large arches.

We shall express uniformly by Q the horizontal thrust of the arch, or the pressure which the two halves exercise on each other, and it will be seen, p. 15 (Theory), that this expression is determined by the condition to be the greatest possible force that must be applied horizontally in N, to prevent a portion *mn*NM of the vault to turn from top to bottom on the edge *m*. Let the co-ordinates AO, MO of the point M, be *a* and *b*, A being the origin, the length of the joint MN=*c*; and *x* and *y* the co-ordinates of the point *m*; the length of the joint *mn*=*z*; and the angle formed by this joint with the vertical, $\theta$. The weight of the portion of the arch *mn*NM will be represented by G, as well as the parts of the construction which



it supports, and $\alpha$ will express the horizontal distance from the centre of gravity of this weight to the point A.

On these premises it will be clearly seen that by admitting the exact value of the horizontal thrust Q, calculated as above, we shall easily ascertain the perpendicular pressure that acts on any joint whatsoever *mn*. In fact, this pressure is nothing but the result of the forces Q and G, that influence the portion of the vault *mn*NM, decomposed perpendicularly at *mn*; that is to say, that it is expressed by

$$G \sin. \theta + Q \cos. \theta.$$

This expression for the vertical joint MN, is reduced to Q; and if the first joint AB is horizontal, it would be reduced for this joint to G, the quantity G being then the total weight of the half-arch.

On this occasion two remarks are necessary. The first, that it cannot generally be conformable to truth to calculate the horizontal thrust Q on the supposition of a force applied to the edge N; for the key-stones at the vertex of the arch press against each other on the whole elevation, or at least on a portion of the elevation of the plane of the joint. The pressure being thus subdivided over a certain space under the edge N, it acts, in order to prevent the descent of the portion of the vault *mn*MN, with the arm of a lever less than is supposed, when it is considered as being applied to N; and consequently, we find by this supposition, a value of Q

less than the real one. No sensible inconvenience however in general results from this discovery, as far as regards the equilibrium of the arch, but some may be felt with respect to the exact appreciation of the forces to which the stone-work is exposed.

The second remark is, that even when we arrive at an accurate knowledge of the value of the normal pressure operating on every joint, we cannot deduce from it the effort exerted on the stone-work, as we are ignorant of the manner in which this pressure is subdivided over the surface of the joint. Far from being able to admit that the pressure is equally spread over all this surface, it is known, on the contrary, that over all the arch, with the exception of a small number of joints, the pressure is principally exerted near one of the ends. This circumstance occurs chiefly at the vertex, where the pressure is felt near the upper end of the joint; on the joints of rupture placed in the haunches of the arch, where it acts near the lower ends; and finally, in the lower joints, under the springings, where the pressure manifests itself near the external arris. We must here suppose, conformably to general experience, that the lower parts of the arch have a tendency to be forced outwards.

The manner in which the pressure is spread over the surfaces of the joints is more uncertain, for this reason, that it depends on the precautions with which the beds of the arch-stones are wrought and disposed, on the distribution of the packing, the consistence of the mortar,

on the extent of the settlement of the arch, according to which the joints are more or less open, &c.

In order, however, to present some slight sketches on this subject, let us suppose an arch built of hard stone shall be wrought and set without mortar; suppose, moreover, that the lower part of the arch has scarcely substance and weight enough to resist the thrust, so that the equilibrium is nearly interrupted by the sinking of the upper, and the reversal of the lower parts. In that case, when we consider, in the first place, the joint MN, placed at the vertex of the vault, we may admit that if this joint were on the point of opening, the pressure at M would be nothing. We may suppose, moreover, that the stone admits of being compressed in a slight degree; this compression, which is almost imperceptible at the lower part of the joint, increases progressively from the point M to N, where it is at its maximum, and in the same way that the pressures experienced by the different portions of the surface of the joint increase also uniformly from the lower end M, where the pressure is nothing, to the upper end N, where it is in the greatest possible degree. Then still preserving the aforesaid denominations, let K be the value of the pressure at its maximum, which takes place at the upper end N, this value being referred to a unit of surface; the pressure exercised at the distance $v$ from the lower edge or arris M, will be $K\frac{v}{c}$, and the pressure exercised on the element $dv$ of the height of

the voussoir will be $K\frac{\nu d\nu}{c}$. It is besides to be observed, that the condition which determines the amount of the pressure exercised on all the elements ought to be such, that the sum of the moments taken with regard to the arris $m$, will be equal to the moment of the weight G of the portion of the arch $mn$NM, taken with regard to the same arris.

The equality of the sum of these moments may be thus expressed:

$$\frac{K}{c}\int_0^c d\nu.\ \nu(b-y+\nu)=G(\alpha-x);$$

or, by integration,

$$\frac{1}{6}K[3(b-y)c+2c^2]=G(\alpha-x).$$

$$\therefore K=\frac{6G(\alpha-x)}{3(b-y)c+2c^2}.$$

This formula will give, on the hypothesis which we have admitted, and which is not far from the truth, for an arch constructed as above, the value of the greatest pressure supported by the parts of the joint MN placed at the vertex of the arch. It is moreover evident, that as this value of K varies with the position of the joint $mn$, where the rupture is supposed to take place, it will be necessary to ascertain the position of this joint, which will give the greatest possible value for K.

The expression for the horizontal thrust, when cal-

culated on the supposition that it operates solely against the edge N, is

$$Q = \frac{G(\alpha - x)}{b - y + c}.$$

According to the data assumed above, the expression of this thrust is the sum of the elementary pressures $K\frac{\nu d\nu}{c}$, taken from 0 to $c$, that is to say, $\frac{1}{2}Kc$; or substituting the value of K we have,

$$\frac{3G(\alpha - x)}{3(b-y)+2c},$$

which is a value somewhat greater than the preceding.

With respect to the joint $mn$, which we shall suppose to be the joint of rupture, that is to say, the one which coincides with the greatest value of K, we shall be allowed equally to admit (since this joint is supposed to be ready to open), that the pressure is nothing at the upper edge $n$, and that it increases uniformly, setting out from this edge, till it reaches the lower edge $m$, where it is in the greatest possible force. If again we designate by K the maximum of the pressure with reference to a unit of surface, and by $\nu$ the distance of any point whatsoever of $mn$ from the extremity $n$, we shall have $K\frac{\nu}{z}$ for the pressure which takes place in this point, as we have taken $z$ to be the length of the joint. The sum of the pressures operating on the whole surface of the joint will then be

$$\frac{K}{z}\int_0^z d\nu.\nu, \text{ or } \frac{1}{2}Kz.$$

As we have given, at p. 61, an expression for the effort directed against the joint, by making it equal to that amount, we shall get

$$\frac{1}{2}Kz = G \sin. \theta + Q \cos. \theta,$$

$$\therefore K = \frac{2(G \sin. \theta + Q \cos. \theta)}{z}.$$

This value of K, or the greatest pressure experienced by the parts of the joint of rupture *mn*, is precisely the double of the result obtained from supposing the effort equally divided over the whole surface of the joint.

The same considerations may be applied to the lower joint AB, which is ready to open in A, when the equilibrium of the vault is on the point of being interrupted. In order to calculate the value of K, which belongs to this joint, we may use the foregoing formula, in which G will represent the whole weight of the half-arch; and if the joint is horizontal, we shall have simply

$$K = 2G.$$

Experience has fully proved that in arches similar to those usually adopted in modern bridges, not only the joints placed at the vertex, and at the points of rupture in the haunches, but also several joints near them, open when the equilibrium is once interrupted. In that case, the preceding formulæ are applicable to them. With respect to the intermediate joints, which have no tendency to open, it would not be more conformable to truth to suppose the pressure to be nothing at either of

the extremities of the joint. If we set out from the vertex of the arch, in going towards the haunches we pass the joints where the pressure tends to operate solely on the upper and outer arrises to those where it tends to operate solely on the lower and inner ones. We should find in the interval a joint in which the pressure does not tend to exert itself more on one end than on the other. This is determined by the condition that the direction of the one resulting from the horizontal pressure Q, and the weight of the portion of the half-arch that is above this joint, passes at an equal distance from the upper and lower ends. It may be supposed, with respect to the joint in question, that the pressure is equally spread throughout its whole depth. We should find, in like manner, a joint subject to the same condition, between the joint of rupture in the haunch, and the joint at the springing of the arch. It is obvious in this view, that if we apply to these intermediate joints the formula specified above, to determine the maximum value K of the pressure supported by the stone-work, we should have a value greater than the true one; so that these formulæ give, at least for the latter joints, the knowledge of a limit which the pressure cannot go beyond.

All the foregoing observations are, moreover, founded on the supposition that the parts of the arch are in a state of equilibrium which is near to being destroyed, so that the joints are almost in the act of opening at the points of rupture. But in reality, there will always be a surplus of resistance, for if the arch-stones may be supposed to have been well wrought and placed in contact, so as

to render bulging and change of form impossible, the joints will not open. In this state, it may be determined that there will be really no pressure upon those of the joints liable to rupture which are the least acted upon, as has been supposed above. We may therefore conclude that in calculating the value of K by the foregoing formulæ, we shall discover limits which the real pressure cannot pass, and from which it will be the more remote, as there will be a greater surplus of resistance in the lower parts of the arch.

If however we suppose the arch-stones to be set with packing and in mortar, or in other compressible materials which may admit of settlement and change of figure in the arch, and permit the joints to open or close, the hypotheses and the foregoing calculations will be at variance with the natural effects. It happens, in fact, that in consequence of the gaping of the joint at one of its extremities, and its binding at another, especially near the points of rupture, the arch-stones are entirely separated throughout a portion of their depth, and only bear upon one another at and near the end where the stress takes place. This effect was observed on striking the centres of the Bridge of Nemours, where the first arch-stones bore upon the impost only about $0^{m}{\cdot}32$ (1·05 feet) of the depth. The layer of mortar with which the joint had been filled before the centering was struck adhered to the bed of the impost, and was separated from that of the arch-stone on the remainder of the joint, the whole depth of which was $1^{m}{\cdot}30$ (4·265 feet).

It is difficult, in such cases, to arrive at an accurate

notion of the action to which the stone is subjected, and to establish a parallel between this action and that which it supports in experiments upon small cubes of the same material. Calculating upon the results of such experiments, it would appear that if the bearing were confined to the proportion above supposed of the surface of the bed of the arch-stone, there would be no occasion whatever for uneasiness as to the result.

It may be remarked also, that even if precautions are taken to adapt the joints to the probable results of settlements, by setting those open which are likely to bind in settling, and by setting the joints close which are likely to open, it does not follow that the pressure will so distribute itself as to produce the desired effect.[1]

It may be stated besides, that in diminishing the depth of the stones composing an arch, the resistance which the arch-stones oppose to pressure is not diminished in the same degree, especially if the diminution in that respect tends to increase the length of the radius of the intrados without lowering the vertex of the arch. In fact, by reducing the substance of the arch, the weight with which the upper part is charged is reduced, at the same time, in a considerable degree.

If the vertex of the curve of extrados is not depressed, then no change is made in the arm of the lever by which the pressures exercised against each other,

[1] The observations, of which the foregoing is the substance, have reference to a mode of practice which is not pursued in this country, and which, literally translated, would not be understood.

by the two halves of the arch, support the superincumbent weight; and even in the case of an arch which is the segment of a circle, or of one composed of various curves, this arm of the lever is increased, because the effect of a diminution of weight in the upper part of the arch is to depress the joints of rupture. Such being the fact, we need not be surprised to find a considerable difference between the degrees of thickness allowed by different architects to arches having the same dimensions, and again, to the diversity that is found with respect to the resistance which various sorts of stones offer to pressure. For though, in general, stone that offers the most resistance is likewise the heaviest, the difference found between the degrees of resistance of which various sorts of stone are capable, is much greater than that which exists between their specific gravities respectively; so that with harder stone the same solidity may be obtained with much less substance.

## XI.—ON THE THEORY OF ARCHES, AND OBSERVATIONS ON THE FUNDAMENTAL PRINCIPLES ON WHICH IT IS GROUNDED.

In the works of Perronet we find a description of the effects which manifested themselves in the large arches which were constructed under his direction, both whilst the arch-stones rested upon the centres, after the key-stones were set, and when the centering had been

struck. He has pointed out methods of striking the centering so as not to subject the curvature of the arches to alteration, and noticed what precautions are necessary to be taken in setting the voussoirs. All the remarks which he has published constitute a consistent system, and throw much light on this subject.

It may be generally remarked, he observes,[1] that the lower courses of arch-stones may be set without the assistance of centering, which becomes necessary only when they begin to slip upon one another, and this usually happens when the beds of the joints form with the horizon an angle of about 40 degrees. At and after that point, the centres begin to support a portion of the weight of the arch-stones, they spread at the base, and where a reversed centre is adopted, it rises at the vertex, unless hindered from doing so by the application of a weight more or less considerable.

The arch of St. Edmé, at Nogent-sur-Seine, appears to be the work for which these effects have been observed and described with the greatest care. Its form is a compound, with a span of $29^{m}{\cdot}24$ (95·9 feet), and a rise of $8^{m}{\cdot}77$ (28·77 feet), the depth of the key-stones being at the vertex $1^{m}{\cdot}62$ (5·31 feet). Each half of the arch is composed of 47 radiating courses of stones, without including the key-stone. The first twenty voussoirs being placed, the five last separated in consequence of

[1] Works of Perronet. Essay upon setting and striking the centres of bridges.

the settling of the centering on which they were placed; the joint opened 20 millimetres (·7874 inches) at the extrados above the 15th course, and there was a vertical disjunction between the arch-stones and the horizontal courses of the spandrels, the effect of which was felt at the seventh course. As the structure advanced these joints closed, and the point of separation between the acting and resisting points having been carried higher by the addition of a greater number of arch-stones, the joints opened at the extrados about 2 millimetres (·0787 inches) from the 26th to the 31st course.

In the bridge of Neuilly, the arches adjacent to the abutments, which are composed of fifty-six radiating courses on each side of the key-stone, the joints successively opened at the extrados, as the setting of the arch-stones advanced, from $\frac{1}{2}$ millimetre (·0197 inches) to 5 (·1968 inches) and 7 millimetres (·2756 inches), from the 11th to the 36th course. Similar effects have been witnessed in several other bridges.

After the key-stones are placed, the effects produced by the weight of the arch-stones show themselves in a different manner. The centres, which before had been loaded in the lower part, giving the vertex a tendency to rise, are now loaded at the summit, and tend, in consequence, to rise in the haunches.

In the arches of the bridge of Neuilly it was observed that when the last joints which had opened at the extrados closed, new joints began to open at the intrados, commencing at the key-stone on both sides,

from $\frac{1}{2}$ millimetre (·0197 inches) to 2 millimetres (·0787 inches), and extending from the 11th to the 36th course. In the bridge of Nogent, the vertical disjunction which had taken place between the arch-stones and the courses of the spandrels almost entirely disappeared, and the last joints that opened at the extrados, in the upper parts of the arches, also closed up.

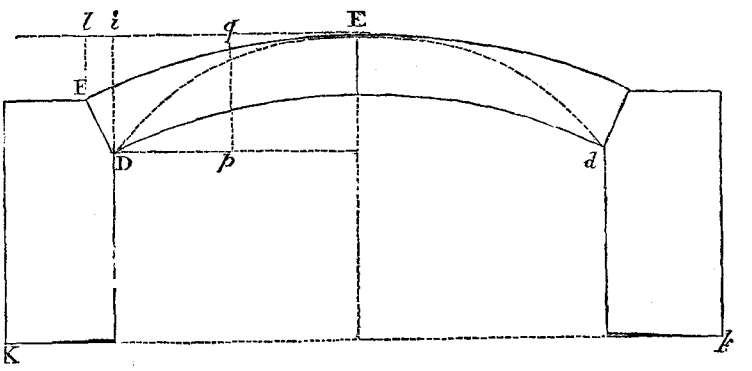
Before the centering of this last mentioned bridge was struck, three straight lines were drawn on the heads of the arch-stones, corresponding from the head of the 28th arch-stone on one side, to the head of that on the other; the other two were inclined and drawn over the haunches of the arch from the extremities of the first line to the points where the 7th course meets a tangent vertical to the springings of the arch. The positions of the extremities of these lines were considered to be fixed points, and the object was to ascertain, by means of the change that might be perceptible in their position and in their form, what would be the play or movement of the arch-stones during the settlement of the arch.

The curvature of the upper line indicated a vertical settlement, which went on uniformly diminishing from its middle to its extremities. As for the other two lines, they formed in their curvature a point of inflexion at the meeting of the joint of the 16th and 17th courses, which indicated, besides the vertical settlement, and the closing of the joints in the upper courses, as far as and comprising the 17th, a similar closing up in the joints of the inferior part, which was, besides, found to be thrust towards the abutments.

It is a natural conclusion, from these observations, which may be repeated on all constructions of the same kind, that the upper part has no tendency to drive back the lower parts by sliding upon the joints of rupture, as was supposed according to La Hire, and, consequently, that the results derived from calculations upon that hypothesis must be erroneous. To obtain an accurate idea of the nature of the thrust, it is necessary to consider successively the two principal epochs in the construction of an arch.

When most of the arch-stones are set, and are approaching towards the key-stone, the centre is loaded considerably, as it supports the whole weight of the arch, and it suffers in consequence an action which is especially felt towards the vertex. Every stone tends downward in proportion to its proximity to the key-stone, and it is clear that any stone in descending must turn on its lowest axis, whence the joint must open at the extrados. This separation is especially sensible at the point where, because of the inclination of the beds of the joint compared with the direction of the weight, the vertical action distributes itself more unequally over the subjacent arch-stones, and it is for this reason that, in the bridge of Nogent, the most open joint was found to be placed near the 26th course.

When the key-stone is set, and the centering struck, the upper parts of the arch DE and $d$E are no longer supported but by their mutual pressure, and, by reason of the settlement that takes place, their point of common support is of course carried to E at the




extrados: the joints then tend to close up, as has been constantly observed, and some architects have endeavoured to add to this effect by driving wedges, the object of which is to augment the solidity of the arch, at the same time that the energy of the pressure which these two parts of the arch direct against one another, and by means of which they are mutually supported, completes the compactness of the structure.

The effort of this pressure is however necessarily carried towards the abutments, and the lower parts of the arch, which it has a tendency to throw out by making them turn upon their exterior arrises K and *k*. Each half of the vault is separated into two parts, at certain points D and *d*, which serve as points of support to the upper parts, and by means of which their action is transmitted to the abutments; these points of support are necessarily found at the intrados. If the abutments have not stability enough to resist the action of the arch, the four parts will fall, turning upon the points K, D, E, *d* and *k*. If they have power to support it, the effect of the settlement will be confined to closing up the joints at the extrados near the point E, and at the in-

trados near the points *d* and D, and to make them open at the intrados near the point E, and at the extrados near the points D and *d*.

The position of the points *d* and D, which have been termed points of rupture, and which it is extremely important to know accurately, depends on the figure of the arch and the distribution of the weight which it supports. In the bridge of Nogent, of which we have spoken above, the position of the joints liable to rupture was naturally indicated, between the 16th and 17th courses of arch-stones, by the points of inflexion on the two lower lines drawn on the heads of the arch-stones. It was not possible to discover it by the same means in the bridge of Neuilly, by reason of the form of the heads of the arch-stones, which are in an arc of a circle, but it was found that the points of rupture were placed between the 26th and 27th courses, because it was in that spot that the joint opened most at the extrados.

It is obvious that arches can sink only in the same proportion that the stones nearest to the key-stone, to the points of rupture, and the base of the abutments, detach themselves one from another, by turning on their upper or lower arrises. The tenacity of mortar may oppose itself to this effect, and that tenacity may be thus sufficient to hold the arch together, as we sometimes find in ancient structures, when the piers or abutments have not the power of resistance which they should have. At the same time, the adhesive power of mortar must not be trusted to, because it does not take effect until after a lapse of time; and although by letting the centering

remain, we may afford the masonry time to become perfectly firm, it is not advisable to reckon upon any degree of strength to be acquired by that means, and which it would be difficult, besides, to compute with any degree of certainty.

The preceding remarks are fully confirmed by the observations which we have made on several arches that were in danger of falling, and on those which we have ourselves had occasion to take down. With respect to the latter, we took care to have horizontal trenches carefully formed in their piers, and we have uniformly observed that the first disjunctions appeared at the intrados near the key, that others were formed afterwards towards the haunches, where their greatest breadth was at the extrados, and that, in short, the upper part was depressed by dividing itself into two principal parts, each of which overturned the pier that it had stood upon.

This theory is equally in unison with the direct experiments which we undertook to make on this subject. We constructed arches in the full semicircle, and in the flat-vaulted arcs lowered $\frac{1}{3}$ and $\frac{1}{4}$, and others in segments of a circle. Their span was 65 centimetres (25·59 inches); the voussoirs, 27 millimetres (1·063 inches) in breadth at the inside, were made of wood, and cut with the utmost exactness. We endeavoured to destroy the equilibrium between the thrust of the upper parts, and the stability of the lower ones, both by diminishing the substance of the piers, and by loading the summit of the arch, and we have constantly remarked

that the rupture had a tendency to operate after the manner which we have described.

These experiments have been repeated on a larger scale by M. Boistard.[2] The arches which he employed were constructed with great exactness with voussoirs of bricks wrought as stones, the thickness and height of the joints of which were 109 millimetres (4·29 inches). The span of the arches was $2^{m}\cdot274$ (7·46 feet), and their length $0^{m}\cdot22$ (8·66 inches).

Arches varying in form from that of a semicircle to almost perfect flatness have been made with these materials, the versed sines of which were respectively $\frac{1}{4}$, $\frac{1}{9}$, and $\frac{1}{17}$ of the aperture. They were constructed with a centering, and their rupture occurred upon the centering being let down vertically ; whether it was occasioned by loading the crown of the arch, or by diminishing the thickness and strength of the abutments.

Each of the arches which we have mentioned was subjected to three principal trials. In the first, the voussoirs were cut down to about 108 millimetres (4·25 inches) in depth, and, as this substance was not sufficient to enable them to support themselves when the centering was struck, a certain number of the voussoirs of the upper part sunk in a vertical direction with it, and made a lodgement on its summit. The two lower parts of the arch then produced the effect of two salient arcs, and divided themselves into two portions. The

[2] See a work entitled *Recueil d'expériences et d'observations faites par M. Boistard.*

last joints of each part opened at the intrados, near both the upper extremities, and the springings of the arch, and the rupture had a tendency to show itself towards the middle, where the voussoirs did not touch the centering, and where the joints opened at the extrados.

In the second trial, in which the voussoirs were again cut down, a certain number of them in the lower part were embraced by a cord that was drawn over the extrados, and which was kept stretched by a weight. The pressure which this cord produced over the last voussoirs counteracted the tendency to separate, which the upper parts of the arch produced; and it has been constantly observed—1st, that if the weights that produced the tension of the two cords were not sufficient for the equilibrium, the arch broke, opening at the intrados near the key-stone and near the springings, where the last voussoir tended to swing round upon its exterior arris, and at the extrados in the haunches; 2ndly, that in case the weights were sufficient to maintain the equilibrium, the same joints opened again in the same manner, in consequence of an unavoidable movement, but the action of the weights had a tendency to bring them together again, and they opened and shut alternately by a sort of oscillating motion, during which the parts of the arch turned alternately in the two directions round the points of support which the arrises of the consecutive arch-stones presented to them; 3rdly, that finally, when the tension of the cord was so great that the pressure it exercised on the lower parts of the

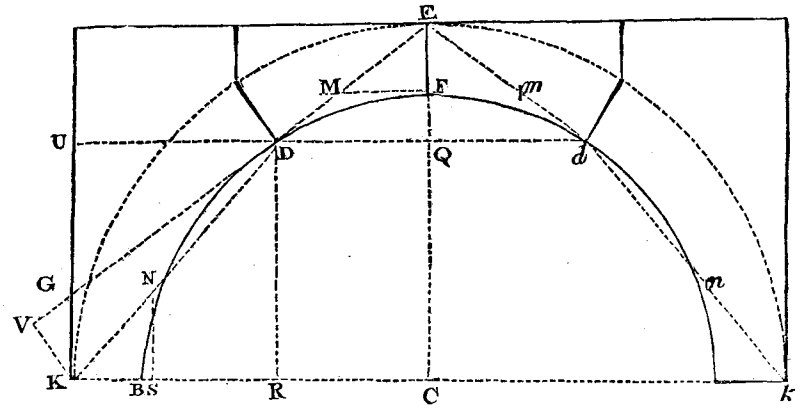
arch was capable of making the upper parts shoot up, the same effects were displayed in an opposite direction, that is to say, that the arch burst at the key, where it opened at the extrados, at the haunches, where it opened at the intrados, and at the springings, where the last voussoir turned round on its inner arris. When the arches were raised on piers, the effects were precisely the same, except that the piers acted with the lower parts of the arch, which, in falling, tended to turn round the outer arris of the base of the piers, instead of turning round that of the voussoir placed at the springings.

In the third trial, buttresses were raised, and the spandrels of the arch were filled up with masonry to a level with the vertex, where the arch was still 108 millimetres (4·25 inches) in depth. When the stability of the abutments was sufficient to resist the thrust, the arch retained its primitive figure after the centering was struck. When the weight on the upper part was increased, the arch broke in the usual way, and the position of the joints of rupture in the haunches was determined by the weight, and the manner in which it was distributed at the crown of the arch, and by the height of the piers on which this arch was sometimes raised. It was generally observed that when the arches were not raised on piers, the rupture had a tendency to appear at an angle of about 30 degrees of the semicircle describing the semi-arch, or towards the angle of 50 degrees of the small arc, in the flat-vaulted arch described with three arcs of a circle. The point of rupture usually

shoots upwards when the height of the piers is increased, and where the crown of the arch is overloaded.

All these experiments, which we regret our inability to give more in detail, completely confirm the principles which we have laid down as to the nature of the settlements of arches.[3]

The result is, that in investigating the principles of their equilibrium, we may (without any material error) consider them as systems of four levers KD, DE, E*d*, *d*K, each loaded with the respective weight of the parts




of the vault that correspond with it, and capable of turning round the points of support K, D, E, *d*, *k*, where they are connected by hinges, as it were: the position of the points *d* and D depends, for each parti-

[3] We find in the *Etudes relatives à l'art des constructions*, published by M. Bruyère, inspector-general of roads and bridges, first edition, the details of some experiments made about the year 1784 by M. Lecreulx, on the occasion of the construction of the bridge of Frouart. They lead to the same results as those specified above, and are particularly remarkable for ascertaining the effect of the division of a mass of abutments into many parts.

cular cas | on the form of the arch, and the distribution of the weight with which it is loaded.

The same principle holds good with regard to arches in the form of the arc of a circle, and to plate-bandes or flat arches, where the vault and its abutments form a system precisely similar. But it is here necessary to remark, that, on account of the shape of the arch, the position of the points of rupture is situated at the springing, unless the vault happens to have the versed sine of the arc described, but small compared with the radius. The experiments made by M. Boistard prove that an arc whose versed sine is $\frac{1}{4}$ of the chord has its joints of rupture placed at the springings, where the haunches of the vault are filled up with masonry.

The calculation applied to the foregoing theory is clear of all difficulties. The points N, M, $n$, $m$, being those in which the levers are met by verticals which pass through the centres of gravity of the corresponding parts of the vault, we may suppose that these levers are loaded at those points with four weights equal to those of these parts, and it will be requisite to determine the relation which must exist between the weights and the direction of the levers, in order that the equilibrium may be maintained.

Let us only consider, for the sake of more simplicity, the half of the vault, which is divided into two symmetrical parts by the axis EC, and let us suppose, at first, that the point D is fixed; and let $\mu$ be the weight applied at M. No change will be made in this system by substituting two other weights; the one

applied at E, and represented by $\mu.\frac{FQ}{EQ}$,* and the other applied at D, and represented by $\mu.\frac{EF}{EQ}$. The point E will then be loaded by a weight equal to

$$2\mu.\frac{FQ}{EQ},$$

and there will result from it, in the direction of the lever ED, a pressure represented by

$$\mu.\frac{FQ}{EQ}.\frac{ED}{EQ},$$

and which, if the point D were really fixed, would be destroyed by its resistance, as well as the effect of the vertical force $\mu.\frac{EF}{EQ}$ which is applied at the same point. But as this point D is situated at the extremity of another lever, which has its point of support in K, and which is loaded at the point N with a new weight, which may be represented by $\nu$, it will be necessary, in order to maintain the equilibrium, that the sum of the moments of these different forces, taken with reference to the point K, be nothing, which will give the equation

* Let W = weight at E, and $\omega$ the weight at D; then $W+\omega=\mu$,

$W+\omega : W :: DE : DM$

$W+\omega : \omega :: DE : EM$

but $DE : DM :: EQ : FQ$

and $DE : EM :: EQ : EF$

$\therefore W+\omega : W :: EQ : FQ$

$W+\omega : \omega :: EQ : EF$

$$W=\frac{(W+\omega)FQ}{EQ}=\frac{\mu.FQ}{EQ}$$

$$\omega=\frac{(W+\omega)EF}{EQ}=\frac{\mu.EF}{EQ}.$$

$$\mu.\frac{FQ}{EQ}.\frac{ED}{EQ}.KV=\mu.\frac{EF}{EQ}KR+\nu.KS,$$

KV being a perpendicular let fall from the point K on the line ED produced, KS and KR the horizontal distances from the point K to N and D.

It will then be seen that

$$KV=\frac{KU\cdot DQ-DU\cdot EQ}{ED},*$$

and substituting this value in the preceding equation, it becomes

$$\mu.\frac{FQ}{EQ}.\frac{DQ}{EQ}KU=\mu.KR+\nu.KS;$$

and if we wish the system to possess stability, we must have

$$\mu.\frac{FQ}{EQ}.\frac{DQ}{EQ}.KU.<\mu.KR+\nu.KS.$$

It is plain that the value of the thrust, and consequently the thickness that must be allowed to the piers, increase with the value of FQ, that is to say, when the centre of gravity of the upper parts of the vault approximates to its vertex. It is the same for the horizontal pressure which the two parts of the vault

* The triangles GKV, DGU and DEQ, are evidently similar

$$KV : KG :: DQ : ED$$

$$\therefore KV=\frac{DQ\cdot KG}{ED}=(KU-GU)\frac{DQ}{ED}.$$

$$=\frac{KU\cdot DQ-GU\cdot DQ}{ED},$$

but $DQ : EQ :: DU : GU$

$$DQ\cdot GU=DU\cdot EQ$$

$$\therefore KV=\frac{KU\cdot DQ-DU\cdot EQ}{ED}.$$

exert upon one another in the case of the equilibrium, as is clear from the form of its expression, which is evidently

$$\mu.\frac{FQ}{EQ}.\frac{DQ}{EQ}.$$

We agree with the remark of M. Boistard, that a mistake has hitherto prevailed by seeking to make the sinking of a vault depend on the diminution of the length of the curve of the intrados. It evidently depends on the shortening of a curve $DEd$ (see figure, page 75), which joins the points by which the arch-stones lie on each other near the joints of rupture, and which may be considered also as joining the points of support of the intermediate arch-stones. The nature of this curve is not easily determined, but it may be considered, in its application, without any sensible error, as being of the same species as the curve of the intrados.

## XII.—APPLICATION OF THE THEORY TO DETERMINE THE THICKNESS OF PIERS AND ABUTMENTS.

The equation $$\mu\frac{FQ}{EQ}.\frac{DQ}{EQ}.KU=\mu.KR+\nu.KS$$

contains all that is necessary to resolve the question which makes the subject of the present section, and the only point is to find a value of BK that will satisfy this equation, or rather which shall render the second member greater than the first, so that the arch may have sufficient stability. That, however, supposes that the position of the points of rupture of D and $d$ is

known *à priori*, a circumstance which does not generally occur, but which it is necessary previously to determine.

In order to do so, we must observe that the points should be so placed, that the momentum of the force that tends to overturn the lower part should be the greatest possible in proportion to the forces that tend to retain it in its position. It is necessary therefore to seek for a value of the arc BD that corresponds with the *maximum* of the expression

$$\frac{\mu . \frac{FQ}{EQ} . \frac{DQ}{EQ} . KU}{\mu . KR + \nu . KS.}$$

The calculation is nearly impracticable for arches forming a complete semicircle by reason of the transcendental qualities which the nature of the circle introduces into this formula, and it becomes entirely so for arches that are composed of several arcs. We must, therefore, have recourse to an indirect method, which consists in assuming different positions of the point D, and determining for each the corresponding value of BK. We may be guided in this species of guess-work by the example of well known bridges, the form of which resembles that of the one in question. It is clear, moreover, that the greatest value assigned to BK will be one that must be taken into consideration, and that the position of the points of rupture will be determined by the corresponding value of the arc BD.

The following table contains the results of this calculation for the vaults that are most frequently in use. We must suppose their span to be $20^m$, (=65·618 feet),

the thickness at the key to be one metre, (=3·2809 feet), and the extrados level.

| Nature of the arches. | Thickness of the abutments. | Position of the points of rupture. |
|---|---|---|
| | Metres. Feet. | Degrees. |
| Semicircular . . . . . | 0·45 = 1·476 | 27 |
| Flat arches $v.s = \frac{1}{3}$ span . . . . . | 0·66 = 2·165 | 45 |
| Do. Do. $\frac{1}{4}$ do. . . . . | 0·82 = 2·690 | 54 |
| Circular arc of 60 degrees on piers 5 metres (= 16·4 feet) high . . . . | 2·95 = 9·679 | 0 |

*Note.* The numbers of the degrees shown in the third column are counted from the abutments, and on the small arc in the flat arches, supposing them to be described with three arcs each equal to the sixth of the circumference.

The results contained in the above table are much under the ordinary dimensions, and the theory which we have explained points out degrees of thickness less considerable than those hitherto allowed to the piers of bridges.

On this subject it is proper to observe that the foregoing calculations suppose that the different portions of the vaults form solid masses, all the parts of which are perfectly connected, and not subject to any sinking. They suppose, likewise, that the abutments are built upon a foundation entirely incompressible, and that, in the fall of the arch, these abutments would turn, without any disjunction, round their exterior edge. These suppositions are generally far from truth. The fall of a bridge cannot occur without some disjunctions in the abutments, with whatever care they may have been constructed; and even should none occur, the abutments could not turn round their exterior edge, where the stones would necessarily be crushed under the pressure

they would have to support, which pressure it should be our business, for this reason, to subdivide over a sufficiently extensive surface. With respect to the incompressibility of the foundations, this condition is very difficult to be accurately ascertained, especially when the masonry is not placed on a platform resting on piles; and it is clear that the downfall of the greatest number of bridges ought to be attributed to the failure of the supports. But as the foundation is the more incompressible as the pressure it supports is distributed over a wider surface, the architect is compelled to increase the dimensions of the points of support in order to adapt the physical circumstances to the analytical hypothesis.[1]

[1] It appears that the experiments of Boistard gave rise to this new theory here treated of by Gauthey, and which Colonel Audoy has shown to coincide exactly with that of Coulomb, as far as rotation is concerned. The theory of this celebrated man is therefore more general, as it takes into account the tendency the voussoirs have to slide on each other; and, besides, it has the advantage of being simpler in its form. See Garidel's Tables des Poussées des Voûtes en Plein Cintre, and Navier's Notes on Gauthey.

# THEORETICAL AND PRACTICAL PAPERS ON BRIDGES.

## I.—ON THE THEORY OF THE ARCH.

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### GENERAL CONDITIONS OF THE EQUILIBRIUM OF A STRUCTURE OF UNCEMENTED STONES.

A STRUCTURE may yield under the pressures to which it is subjected either by the slipping of certain of its surfaces of contact upon one another, or by their turning over upon the edges of one another; and these two conditions involve the whole question of its stability.

Let a structure MNLK, fig. 1, composed of a single row of uncemented stones of any forms, and placed under any given circumstances of pressure, be conceived to be *intersected* by any geometrical surface 1 2, and let the *resultant* $a$A of all the forces which act upon one of the parts M N 2 1, into which this intersecting surface divides the structure, be imagined to be taken.

Conceive then this intersecting surface to change its form and position so as to coincide in succession with all the common surfaces of contact 3 4, 5 6, 7 8, 9 10, of the stones which compose the structure; and let $b$B, $c$C, $d$D, $e$E, be the resultants, similarly taken with $a$A, which correspond to these several planes of intersection.

In each such position of the intersecting surface, the resultant spoken of having its direction produced, will intersect that surface either *within* the mass of the structure, or, when that surface is imagined to be produced, *without* it. If it intersect it *without* the mass of the structure, then the *whole* pressure upon one of the parts, acting in the direction of this resultant, will cause that part to turn over upon the edge of its common surface of contact with the other part; if it intersect it *within* the mass of the structure, it will not.

Thus, for instance, if the direction of the resultant of the forces acting upon the part N M 1 2 had been $a'$ A', not intersecting the surface of contact 1 2 *within* the mass of the structure, but when imagined to be produced beyond it to $a'$; then the whole pressure upon this part acting in $a'$ A' would have caused it to turn upon the edge 2 of the surface of contact 1 2; and similarly if the resultant had been in $a''$ A'', then it would have caused the mass to revolve upon the edge 1. The resultant having the direction $a$A, the mass will not be made to revolve on either edge of the surface of contact 1 2.

Thus the condition that no two parts of the mass should be made by the insistent pressures to turn over upon their common surfaces of contact is involved in this other, that the direction of the resultant, taken in respect to every position of the intersecting surface, shall intersect that surface actually *within* the mass of the structure.

If the intersecting surface be imagined to take up an infinite number of different positions, 1 2, 3 4, 5 6, &c., and the intersections with it, *a*, *b*, *c*, *d*, &c., of the directions of all the corresponding resultants be found, then the curve line *a b c d e f* joining these points of intersection is that to which I have given the name of the LINE OF RESISTANCE.

This line can be completely determined by the methods of analysis in respect to a structure of any given geometrical form having its parts in contact by surfaces also of given geometrical forms. And conversely, the form of this line being assumed, and the direction which it shall have through any proposed structure, the geometrical form of that structure may be determined, subject to these conditions; or lastly, certain conditions being assumed both as it regards the form of the structure and its line of resistance, all that is necessary to the existence of these *assumed* conditions may be found. Let the structure ABDC, fig. 2, have for its line of resistance the line PQ. Now, it is clear that if this line cut the surface MN of any section of the mass in a point *n* without the surface of the mass, then the resultant of the pressures upon the mass CMN will

act through $n$, and cause this portion of the mass to revolve about the nearest point N of the intersection of the surface of section MN with the surface of the structure.

Thus, then, it is a condition of the equilibrium that *the line of resistance shall intersect the common surface of contact of each two contiguous portions of the structure, actually within the mass of the structure;* or, in other words, that it shall actually go through each joint of the structure, avoiding none: this condition being necessary, that no two portions of the structure may revolve on the edges of their common surface of contact.

But besides the condition that no two parts of the structure should turn upon the edges of their common surfaces of contact, which condition is involved in the determination of the LINE OF RESISTANCE, there is a *second* condition necessary to the stability of the structure. Its surfaces of contact must no where slip upon one another. That this condition may obtain, the resultant corresponding to each surface of contact must have its *direction* within certain limits.[1] These limits

[1] The resistance of surfaces is *not exerted exclusively* in the direction of the normal, according to *an hypothesis*, which was probably introduced into the theory of Statics in order to simplify the investigations of those who *originated* that science, but which there seems no reason for retaining any longer. It is exerted in an infinite number of different directions included within a certain angle to the normal, or rather within the surface of a certain right cone, having the normal for its axis and the point of resistance for its vertex. *Any force*, however great, applied within this conical surface, will be sustained by the

are defined by the surface of a right cone, having the normal to the common surface of contact at the above-mentioned point of intersection of the resultant for its axis, and having for its vertical angle twice that whose tangent is the coefficient of friction of the surfaces. If the direction of the resultant be *within* this cone, the surfaces of contact will not slip upon one another; if it be without it, they will.

Thus then the *directions* of the consecutive resultants in respect to the normal to the point where each intersects its corresponding surface of contact, are to be considered as important elements of the theory.

Let then a line, ABCDE, fig. 1, be taken, which is the locus of the consecutive intersections of the resultants $a$A, $b$B, $c$C, $d$D, &c. This line I have called the LINE OF PRESSURE. Its geometrical form may be

resistance of the surface of the mass—and *no force*, however small, without it.

Let R represent a single force, or the resultant of any number of forces applied to a fixed surface, and let R′ and R″ be the resolved parts of R in the directions perpendicular and parallel to the surface. Also let $\rho$ be the inclination of R to the vertical, and $f$ the coefficient of friction. The friction of the surfaces in contact is therefore represented by $f$R′, and motion will, or will not, ensue according as R″ *is* greater or is *not* greater than $f$R′. Or, according as $\frac{R''}{R'}$ *is* greater or is *not* greater than $f$. Or, if $f=\tan.\ \phi$, according as $\tan.\ \rho$ is, or is not, greater than $\tan.\ \phi$, or as $\rho$ is greater or is not greater than $\phi$. In the remainder of this paper the angle $\phi$, or $\tan.^{-1} f$, will be called *the limiting angle of resistance*. This principle of the resistance of a surface was, I believe, first given in my paper on the Equilibrium of the Arch, read before the Cambridge Philosophical Society in December, 1833, and published in the fifth volume of their Transactions.

determined under the same circumstances as that of the line of resistance. A straight line $cC$ drawn from the point $c$, where the LINE OF RESISTANCE $abcd$ intersects any joint 5 6 of the structure, so as to touch the LINE OF PRESSURE ABCD, will determine the *direction* of the resultant pressure upon that joint: if it lie within the cone spoken of, the structure will not slip upon that joint; if it lie without it, it will.

Thus the whole theory of the equilibrium of any structure is involved in the determination with respect to that structure of these two lines—the line of resistance, and the line of pressure. One of these lines, the line of resistance, determining the *point* of application of the resultant of the pressures upon each of the surfaces of contact of the system; and the other, the line of pressure, the *direction* of that resultant.

The determination of both, I have shown, to be, under their most general forms, within the resources of analysis, and I have given general equations for their determination in that case, in which all the surfaces of contact or joints are planes—the only case which offers itself as a *practical* case. The analytical discussion of these equations is reserved for a subsequent paper; my present object is to give a general account of their application to various practical questions of construction. My researches have been confined to that class of prismatic structures, the geometrical form of any of which may be conceived to be generated by the motion, parallel to itself, of a vertical *plane* of given form and dimensions. Such structures are represented in figures 3, 4, 5, 6, &c.; they

include every form of the pier, the buttress, the plate bande, the revetment wall, the stone embankment, and the arch. I have supposed their common surfaces of contact to be planes, and the forces acting on them to be the weights of their parts, and other forces applied from without, but all acting in vertical planes, and applied uniformly along the horizontal breadth of each structure. I have first considered that class of structures (including all those enumerated above, except the arch) in which the planes of section are all parallel to one another. I have determined the general equations to the LINES OF RESISTANCE AND PRESSURE in respect to this great class of structures, and I have shown that when the planes of section are vertical planes, these two lines coincide.

In applying these equations I have first supposed the various external pressures sustained by each structure to be applied only to certain portions of its surface, as for instance its extremities, or the surfaces of a few of the stones near its extremities, and I have thence determined the form and direction of the line of resistance through the rest of the structure. Upon this hypothesis I find the equation to the line of resistance PRST, of the trapezoidal structure, fig. 6, to be (under its most general form) one of three dimensions; showing that if the mass be imagined to be sufficiently extended it will, under certain circumstances, cut the extrados in three distinct points, R, S, T; at each of which points rupture will therefore take place, unless some external resistance be there applied to the extrados.

The highest of these points, R, marks that section which must terminate the structure, that it may stand unsupported under its insistent pressures.

Certain assumed values of the arbitrary constant, which enter into the equation to this line of resistance, convert it into the line of resistance of a buttress, fig. 7, with one or both of its faces AB and DC inclined to the vertical: an inclined pier, figs. 8, 9; an upright pier, fig. 10; or a plate bande in a horizontal or an inclined position, and of an uniform and variable width, figs. 11, 12, 13.

In a pier or wall, figs. 7, 9, 10, of uniform thickness, whose face is inclined to the vertical at any angle, and which has its parallel planes of section inclined to it at any given angle, the line of resistance is always an hyperbola whose magnitude and position are readily determinable in terms of the thickness of the mass and the magnitudes, directions, and points of application of the insistent forces.

In the case of an *upright* pier or uniform wall, fig. 10, the asymptote of this hyperbola is a *vertical* line KE. The pier will not be overthrown by the insistent pressures, however high it may be built, provided this asymptote (determined in respect to those pressures) be found to be (as in the figure) *within* the mass of the pier.

In the case of the uniform plate bande, figs. 12, 13, where the planes of section are all vertical, the line of resistance is always a parabola, and it coincides with the line of pressure. The intersecting planes have, up to this period of the investigation, been supposed to be

parallel to one another. Let this hypothesis now be discarded, and, as the simplest case of a section of variable inclination, let its plane be supposed always to pass through the same horizontal axis. This case includes that of the circular arch under its most general form, and to this case my further researches have been limited.

I have supposed certain forces to be applied to one extremity of a structure thus intersected, and resting by its other extremity upon an immoveable base. As for instance a semi-arch, fig. 14, resting by its extremity B upon its abutments, and supported by a given force P, applied to the key-stone AD, instead of the pressure of an opposite semi-arch. On this hypothesis the equation to the line of resistance may be completely determined in respect to an arch of equal voussoirs subjected to any variety of loading. With a view to this general determination I have first supposed the loading to be collected over a single point X of the semi-arch; and on this hypothesis I have found the equation to the line of pressure in terms, of the inclination of the joint AD of the key-stone (that is, of the line CD) to the vertical, the angle ACB of the segment of the arch, the common depth AD of the voussoirs the point of application, and the magnitude of the force P and the weight X. This determination evidently includes the cases of the loaded Gothic and segmental arches; and were the magnitude and point of application of the force P known, it would constitute a complete determination of the equilibrium of the structure.

But unfortunately, in the actual case of the arch, this pressure upon the key is an unknown thing. We neither know its point of application nor its amount.

It is the pressure of the opposite semi-arch, or rather it is the resultant of an infinity of pressures exerted by the opposite semi-arch upon an infinity of points, by which that semi-arch is in contact with the face AD of the key; and the amount of this resultant, and whether it pass through the middle of the key-stone or its extremities, are necessary, but, up to this period of the investigation, *unknown* elements of the theory. Some other principle of mechanical action manifestly enters into the conditions of the equilibrium, and claims a place at this period of the discussion.

That other principle is this, that of all the pressures which can be applied to the key, different in their points of application and amount, but all consistent with the equilibrium of the semi-arch, that which it actually sustains by the pressure of the opposite semi-arch is the *least*. This condition of minimum pressure at the key supplies mathematically all that is required for the complete determination of that pressure, and perfects the theory.

The demonstration of it is easy. The pressure which an opposite semi-arch would produce upon the side AC of the key-stone, fig. 2, is equal to the tendency of that semi-arch to revolve forwards upon the inferior edges of one or more of its voussoirs. Now this tendency to motion is evidently equal to the least force which would support this opposite semi-arch; supposing the semi-

arches, therefore, to be equal in every respect, and equally loaded, it is equal to the least force which would support the semi-arch ABDC.

Suppose the mass ABDC, fig. 2, to be acted upon by any number of forces among which is the force Q being the resultant of certain resistances, supplied by different points in a surface BD, common to the intersected mass and to an immoveable obstacle BE.

Now it is clear that under these circumstances we may vary the force P, both as to its amount, direction, and point of application, without disturbing the equilibrium, provided only the form and direction of the line of resistance continue to satisfy the conditions imposed by the equilibrium of the system.

These have been shown to be the following,—that it no where *cut* the surface of the mass, except at P, and within the space BD, and that it no where cut any section MN of the mass, or the common surface BD of the mass and obstacle, at an angle with the perpendicular to that surface, greater than the limiting angle of resistance.

Thus, varying the force P, we may destroy the equilibrium, either, first, by causing the line of resistance to take a direction without the limits prescribed by the resistance of any section MN through which it passes, that is, without the cone of resistance at the point where it intersects that surface; or, secondly, by causing the point Q to fall *without* the surface BD, in which case *no resistance* can be opposed to the

resultant force acting in that point; or, thirdly, the point Q lying within the surface BD, we may destroy the equilibrium by causing the line of resistance to cut the surface of the mass somewhere between that point and P.

Let us suppose the limits of the variation of P within which the first two conditions are satisfied, to be known; and varying it, within those limits, let us consider what may be its *least* and *greatest* values so as to satisfy the third condition.

Let P act at a given point in AC and in a given direction. It is evident that by diminishing it under these circumstances, the line of resistance will be made continually to assume more nearly that direction which it would have, if P were entirely removed.

Provided then, that if P *were* thus removed, the line of resistance would cut the surface, that is, provided the force P be necessary to the equilibrium; it follows that by diminishing it, we may vary the direction and curvature of the line of resistance until we at length make it *touch* some point or other in the surface of the mass.

And this is the limit; for if the diminution be carried further, it will *cut* the surface, and the equilibrium will be destroyed. It appears then that under the circumstances supposed, when P, acting at a given point and in a given direction, is the least possible, the line of resistance *touches the interior surface or intrados of the mass*.

In the same manner it may be shown, that when it is the greatest possible, the line of pressure touches the exterior surface or extrados of the mass.

I have here supposed the direction and point of application of P in AC to be given ; but by varying this direction and point of application, the contact of the line of resistance with the intrados of the arch may be made to take place in an infinite variety of different points, and each such variety supplies a new value of P. Among these, therefore, it remains to seek the *absolute* maximum and minimum values of that force.

In respect to the direction of the force P, or its inclination to AC, it is at once apparent that the least value of that force is obtained, whatever be its point of application, when it is *perpendicular* to AC.

There remain then two conditions to which P is to be subjected, and which involve its condition of a minimum. The *first* is, that its amount shall be such as will give to the line of resistance a point of contact with the intrados. The *second*, that its point of application in the key-stone AC shall be such as to give it the least value which it can receive, subject to the first condition.

I have determined the value of P subject to these conditions in a paper read before the Cambridge Philosophical Society in May 1837, and published in the 6th volume of their Transactions. The equations involving that value admit of a complete solution, and determine it for every form and dimension of the broken

or Gothic arch, and the complete segment, and for every circumstance of its loading.

The condition however that the resultant pressure upon the key-stone is subject in respect to the *position* of its point of application on the key-stone to the condition of a minimum, is dependent upon hypothetical qualities of the masonry. It supposes an unyielding material for the arch-stones, and a mathematical adjustment of their surfaces. These have no existence in practice. On the striking of the centres the arch invariably sinks at the crown, its voussoirs there slightly opening at their lower edges, and pressing upon one another exclusively by their upper edges. Practically the line of resistance then, in an arch of *uncemented* stones, *touches the extrados* at the crown; so that only the first of the two conditions of the minimum stated above actually obtains: that, namely, which gives to the line of resistance a contact with the intrados of the arch. This condition being assumed, all consideration of the yielding quality of the material of the arch and its abutments is *eliminated*. It will thus be discussed in what remains of this paper.

To simplify the analytical discussion of the question I have hitherto assumed the load upon the semi-arch to be placed over a single point of it X, fig. 14. I now imagine it to be distributed in any way over the extrados, but symmetrically in respect to the two opposite semi-arches. The centre of gravity of this load on each semi-arch being determined, it is evident that the *horizontal thrust* P on the key-stone of the arch will be the same

if the whole load upon it be imagined to be collected in these two centres of gravity. I determine then the horizontal thrust P on this hypothesis of a concentrated loading: this determination being made, the data necessary to the analytical discussion of the question are *complete*, all the forces acting upon a mass ASTD of the arch and its loading intercepted between the crown and any inclined position CT of the radius are given, and the equation to the true line of resistance under any given circumstances of loading is determinable in terms of the radius vector CR and the angle ACS. The equation determining the value of P is unfortunately one of a high order, involving circular functions of complicated forms; and the solution of it otherwise than by approximation is perhaps to be despaired of. The small value of the ratio of the depth AD of the voussoirs, in the majority of practical cases, to the radius CA of the arch in terms of which ratio the value of P is expressed, suggests a developement of the value of P in a series of terms ascending by powers of this ratio. To effect this developement I have called to my aid the theorem of Lagrange, using two terms only of that theorem, and not therefore extending the approximation beyond the first power of the ratio. It might perhaps be expedient in some cases to extend it to the second; beyond this limit no practical enquiry need however be carried.

The line of resistance being fully determined, the *point* Q, fig. 2, where the resultant pressure of the whole semi-arch intersects, the supporting surface BD

of the abutment becomes known, and also the *direction* of this resultant pressure. Now all the circumstances which determine the equilibrium of an abutment, subject at a given point to a given insistent pressure, I have before discussed, and I have determined its line of resistance under these circumstances: that line of resistance evidently unites with that of the arch at this point—this line of pressure is therefore completely known, and the conditions of the equilibrium of the piers or other abutments of the arch, and of the arch itself, are determined.

I have hitherto considered the form of the solid to be given, together with the positions of the different sections made through it, and I have thence the forms of its lines of resistance and pressure, and their directions through its mass.

It is manifest that the converse of this operation is possible.

Having given the form and position of the line of resistance or of pressure, and the positions of the different sections to be made through the mass, I may for instance enquire what form these conditions impose upon the surface which bounds it.

Or I may make the direction of the line of resistance or pressure and the form of the bounding surface subject to certain conditions not absolutely determining either.

For instance, if the form of the intrados of an *arch* be given, and the direction of the intersecting plane be always perpendicular to it, and if I suppose the line

of pressure to intersect this plane always at the same given angle with the perpendicular to it, so that the tendency of the pressure to thrust each from its place may be the same,—I may determine what under these circumstances must be the extrados of the arch.

If this angle *equal* constantly the limiting angle of resistance, the arch is in a state bordering upon motion, each voussoir being upon the point of slipping downwards or upwards, according as the constant angle is measured above or below the perpendicular to the surface of the voussoir.

The systems of voussoirs which satisfy these two conditions are the greatest and least possible.

If the constant angle be zero, the line of pressure being every where perpendicular to the joints of the voussoirs, the arch would stand even if there were no friction of their surfaces.

It is then technically said to be equilibrated, and the equilibrium of the arch according to this single condition constituted the theory of the arch so long in vogue, and so well known from the works of Emerson, Hutton, Whewell, &c. It is impossible to conceive any arrangement of the parts of an arch by which its stability can be more effectually secured, *so far as the tendency of its voussoirs to slide upon one another is concerned*: there is however, I believe, no practical case in which this tendency really affects the equilibrium. So great is the *limiting angle of resistance* in respect to all the kinds of stone used in the construction of arches that

it would be exceedingly *difficult* to construct an arch, the resultant pressure upon any of the joints of which should lie *without* this angle, or which should yield by the *slipping* of any of its voussoirs.

The theory stated above readily explains the phenomena observed in the settlement of the arch.

In the case of a trapezoidal mass placed in any inclined position and intersected by parallel planes, I have investigated in my first paper (Cambridge Phil. Trans., vol. v. part 3) the equation to the line of pressure, and I have ascertained it to be one of three dimensions, having a point of contrary flexure. This case of a trapezoidal mass thus obliquely placed approaches sufficiently near to that of the arch, to indicate the existence of a similar point of contrary flexure in the line of pressure to the arch. Now it is evident that since the directions of the pressures on the successive joints are all tangents to the line of pressure, this point of contrary flexure in it, shows a change to take place, somewhere, in the directions of the pressures in respect to their inclinations to the joints; the inclinations of the pressures at all the joints above a certain one, being downwards or *towards* the intrados, and those below it upwards or *from* the intrados.

Hence, therefore, it appears that the tendency of the pressure is to cause all the voussoirs above the joint spoken of to slide *downwards*, and those beneath that joint, *upwards;* and that these effects may be expected to follow the striking of the centre of the arch; the

weight being then suddenly thrown upon the voussoirs, and these admitting of a certain degree of motion, in the directions of the forces impressed upon them.

Now this is precisely what was observed at the bridge of Nogent, of the construction of which Perronet has left a detailed account.

Three straight lines were drawn upon the face of the arch before the striking of the centre, one of them stretching horizontally above the crown, and the two others lying obliquely from the extremities of this, towards the springings of the arch.

After the centre had been struck, the lines were observed to have assumed certain curved forms, indicating, in accordance with the theory, a downward motion in all the voussoirs above a certain point on each side, and an upward motion in the voussoirs beneath that point.

These observations have been confirmed by numerous others, and especially by those (made also by Perronet) at the Pont de Neuilly.

Let ABB′, fig. 15, represent an arch having the joints of its voussoirs perpendicular to the intrados as they usually are made. Let RQPQ′R′ be the line of resistance touching the intrados at Q and Q′, and the extrados at the crown in A. The material of the arch may therefore be expected to yield more particularly about the points A, Q and Q′, than any other; a greater proportion of the pressure being there thrown upon the *edges* of the voussoirs.

If by reason of such yielding, or from any other alter-

ation in the forces impressed upon the mass, or in the circumstances of their application, the form of the line of resistance be altered, it may manifestly be expected to intersect the surface of the mass first about *those points;* the least possible alteration of form being there sufficient to produce the intersection. And this being the case, the portion of the arch above Q and Q′ must separate into *two* portions, revolving at those points about the lower portions of the arch (see fig. 16) and at A, upon the extremities of one another.

Nevertheless this revolution is manifestly impossible unless the points Q and Q′ yield outwards. And this can only take place by the yielding of the material at Q and Q′, by the slipping back of the voussoirs there, or by the portions of the arch or its abutments beneath those points revolving outwards, in consequence of the intersection of the extrados by the extremities QR and Q′R′ of the line of resistance (fig. 15).

The last is in point of fact the cause which leads, in the great majority of cases, to the fall of the arch.

The extremity R of the line of resistance is made to cut the extrados of the arch, or the outer surface of the pier, by the diminution or removal of some force which acted there in opposition to the tendency of the arch to spread itself, and which kept the direction of the line of resistance within its mass,—the resistance of a mass of earth for instance, or the opposite thrust of some other arch springing from the same pier or abutment.

On the whole, then, it appears that in the commencement of its fall the arch will, according to this

theory, divide itself into six distinct portions, of which four will revolve about the points S, S′, Q, Q′ and A, as represented in the figure 16. Now this is what is uniformly observed to take place in the fall of the arch.

Gauthey, having occasion to destroy a bridge, caused one of its arches to be isolated from the rest; and, the adhesion of the cement being sufficient to counteract the tendency of the pressure to rupture the piers, he caused them to be cut across. The whole then at once fell, the falling portion separating itself into four parts. Having constructed small arches of soft stone, and without cement, he loaded them until they fell. Their fall was always observed to be attended with the same circumstances. Before the arch finally yielded the stone also was observed to chip at the intrados about the points Q and Q′, round which the upper portions of it finally revolved.

Some experiments made by Professor Robison with *chalk* models were attended with slightly different results. Having loaded them at the crown until they fell, he observed first, that the points where the material began to yield were not precisely those where the rupture finally took place.

This fact presents a remarkable confirmation of the theory expounded in this paper.

It is manifest, that according to that theory, with any variation in the least force P, which would support the semi-arch if applied at its crown, there will be a corresponding change in the position of the point Q.

Now as the load upon the crown is increased, this

least force P is manifestly increased. The result is a corresponding variation in the form of the line of resistance, tending to carry its point of contact with the intrados lower down upon the arch.

This is precisely what Professor Robison observed. The arch began to chip at a point about half way between the crown and the point where the rupture finally took place.

The existence of the points Q and Q′, about which the two upper portions of the arch have a tendency to turn, and about which the material is first observed to yield, has long been known to practical men. The French engineers have named these points the points of rupture of the arch; and the determination of their position by a *tentative method* forms an important feature in the theory which they have applied to this important branch of Statics.

The theory of the equilibrium of the groin and that of the dome are precisely analogous to the theory of the arch.

In the former a mass springs from a small abutment, spreading itself out symmetrically with regard to a vertical plane passing through the centre of its abutment. The groin is in fact nothing more than an arch, whose voussoirs vary as well in breadth as in depth. The centres of gravity of the different elementary voussoirs of this mass lie all in its plane of symmetry. Its line of resistance is therefore in that plane, and its theory is embraced in that which has been already laid down.

Four groins commonly spring from one abutment;

each *opposite* pair being addossed, and each *adjacent* pair uniting their margins. They thus lend one another mutual support, partake in the properties of a dome, and form a continued covering.

The groined arch is of all arches the most stable; and could materials be found of sufficient strength to form its abutments and the parts about its springing, I am inclined to think that it might be safely built of any required degree of flatness, and that spaces of enormous dimensions might readily be covered by it.

It is remarkable that modern builders, whilst they have erected the common arch on a scale of magnitude nearly approaching perhaps the limits to which it can be safely carried, have been remarkably timid in the use of the groin.

I shall terminate this paper by a comparison of the theory developed in it with that of Coulomb, which was unknown to me until my researches had long been completed, and which, after an oblivion of more than sixty years among the pages of the Mémoires des Savants Etrangers, undisturbed by the many and fierce discussions to which the question has been subjected since that time in this country and elsewhere, has recently been exhumed and made the subject of valuable researches by M.M. Navier, Lamé and Clapeyron, and Garidel.

## COMPARISON OF THE PRECEDING THEORY WITH THAT OF COULOMB.

According to the theory of Coulomb (Mémoires des Savants Etrangers, 1773,) as developed by the researches of Navier, (Résumé des leçons données a l'école des ponts et chaussées, 1833, p. 196,) the conditions of the equilibrium of a semi-arch ABNM, fig. 17, in respect to its tendency to revolve upon the edges of its voussoirs, either at the extrados or the intrados, when supported by a force Q acting perpendicular to the joint MN at its crown, are these : any joint *mn* being taken, the force Q must not be less than that which being applied at N would just prevent the portion M*mn*N, if it were a continuous solid, from turning upon the point *m*, and it must not be greater than that which being applied at M would be upon the point of causing it to turn about *n* ; for if the first of these forces be called $Q_1$, and the second $Q_2$, then the force Q being less than $Q_1$ it is evident that if the arch did not turn upon some other point it would certainly turn upon the point *m*, and that this would be the case wherever in NM, Q was applied, since it would be insufficient to the equilibrium when applied in the most favourable position, which is N : again, if Q were greater than $Q_2$, then, provided the arch did not turn upon some other point of the extrados, it would certainly turn upon the point *n*, and that, wherever in MN it were applied, since when applied at the most unfavourable point M, it is sufficient to produce this revo-

lution. The force Q must then be less than one of the forces $Q_1$ and $Q_2$, and greater than the other. These conditions, however, only determine the stability of the semi-arch in respect to the particular joint *mn*; they assure us that rupture will not take place at that joint. Let corresponding values of $Q_1$ and $Q_2$ be found in respect to every joint; then if the value of Q be not less than any one of the series of values of $Q_1$ thus determined, and be not greater than any one of the values of $Q_2$, we are assured that the semi-arch will not turn upon the edges of any one of its joints.

Of all the values of $Q_1$ determined as above one will be the least; that least value equals the least force Q, which will support the semi-arch; it is therefore just equivalent to the tendency of the semi-arch to revolve towards the opposite semi-arch, and thus it is just equal to the pressure of the opposite semi-arch upon it.

This minimum of the forces $Q_1$ is thus, then (according to Coulomb), the actual horizontal thrust upon the key-stone of the arch; it must not exceed the maximum of the forces $Q_2$. This condition is necessary, and it is sufficient to the equilibrium of the arch, so far as the rotation of the voussoirs is concerned.

That particular joint in respect to which the least value of $Q_1$ is obtained is evidently the joint of rupture; it is there that if the horizontal pressure of the opposite semi-arch were rendered less than this value of Q, rupture would take place. Here then (there being no cement) the resultant of the forces upon MN*nm* passes actually through the point *m*, and the line of resistance

passes therefore through that point; so that the point of rupture, as determined by the theory of Coulomb, coincides with that determined by the theory given in the preceding pages. Moreover, the condition of the theory of Coulomb, that the force Q should not be less than one of the forces $Q_1$ and $Q_2$, or greater than the other, is evidently included in this other, that in respect to no joint *mn* of the structure the intersection with that joint of the resultant of the forces on MN*nm* should lie beyond the points *m* and *n*. Now this last condition is identical with that which is the fundamental condition of the equilibrium as determined by the preceding theory, viz., that the line of resistance should cut every joint actually within the mass of the structure. It is apparent that the minimum force $Q_1$ of the theory of Coulomb is identical with the minimum force P of the preceding theory.

So far as the determination of the position of the points of rupture and the amount of the horizontal thrust is concerned, the results of the two theories are the same; the difference is in the principles from which they start and the methods of investigation: that of Coulomb supposes a separate discussion of the conditions of the equilibrium of each particular voussoir, and as many distinct operations of analysis as there are voussoirs, and it determines the required maxima and minima by a comparison of the various different results thus obtained; the theory developed in the preceding pages includes the determination of all these in the discussion, by known methods of analysis, of a *single*

*function*, of the dimensions and form of the arch, and its loading. It here leaves the theory of Coulomb. A *single* analytical relation being established by it between the position of the points of rupture, the amount of the horizontal thrust, the dimensions of the arch, and the form and amount of the loading, these elements may be varied in any way with a reference to one another; the loading, for instance, may be varied in its distribution, so as to vary from one point to another the position of the points of rupture, or so as to vary, from one amount to another, the horizontal thrust.

From the determination of the horizontal thrust, and the position of the points of rupture, it proceeds to the complete determination of the line of resistance throughout its whole length; the condition that this line shall no where cut the extrados or back of the structure establishes the equilibrium in respect to any rotation of the voussoirs on their external edges, and determines the extreme amount of loading, placed on any point of the arch, which it may be made to bear. Traced to the abutment of the arch, the line of resistance ascertains the point where the direction of the resultant pressure intersects it, and the line of pressure fixes the actual direction of that resultant; these elements determine all the conditions of the equilibrium of the abutments, and therefore of the whole structure; they associate themselves directly with the conditions of the loading of the arch, and enable us so to distribute it as to throw the points of rupture into any given position on the

intrados, and give to the line of resistance any direction which shall best conduce to the stability of the structure; from known dimensions and a known loading of the arch they determine the dimensions of piers which will support it; or conversely from known dimensions of the piers they ascertain the dimensions and loading of the arch, which may safely be made to span the space between them.

## CONDITIONS OF THE EQUILIBRIUM OF A STRUCTURE OF CEMENTED STONES.

The condition that the resultant pressure upon any portion of a structure must pass actually *through* the common surface of contact of that portion with the subjacent portion of the structure, on which condition the whole of the theory of the equilibrium of the structure has in the preceding pages been made to depend, is not a condition necessary to the equilibrium of a structure of *cemented* stones.

When the joints of the structure are cemented together, the resultant pressure may have its directions *beyond* the limits of any joint without causing the two portions separated by that joint to revolve upon the edges of one another, the distance of this deviation without the boundary of the joint being of course circumscribed within certain limits, imposed by the adhesive qualities of the cement, the dimensions of the adhering surface, and the inclinations and amounts of

insistent pressures. Thus then that essential condition of the discussion, according to which the line of resistance can no where *intersect* either the intrados or extrados of the arch, but which gives it a *contact* with the intrados at each of the points, called the points of rupture, altogether fails. A very simple consideration however brings this element, of the adhesion of the joints, into the discussion; and the theory is thus made to embrace the whole question of the stability of a cemented structure. It is this, that for the adhesion of the cement at each joint, supposed to operate uniformly over the whole surface of the joint, there may be substituted a force of the same amount acting at a single point on that surface, that point being its centre of gravity; because the adhesion thus uniformly distributed, and acting (so far as the tendency of the stones to turn over on their edges is concerned, which is the only case now under consideration) perpendicularly to the surface of the joint, may be considered to be made up of a system of equal and parallel forces, infinite in number and precisely analogous to the *weights* of the component elements of a thin uniform lamina, of the same form and dimensions as the joint; so that the system of elementary adhesions composing the adhesion of the joint shall have its resultant passing through exactly the *same point* as that through which the resultant of the weights of the elements of the lamina passes, that is, through the centre of gravity of the lamina.

In considering the conditions of the equilibrium of any

portion ASTD of a structure, fig. 14, and supposing the whole adhesive power of the joint ST to be called into operation, and to constitute an element of the equilibrium, we have only then to suppose that adhesion to be collected in the centre of gravity of ST and to act there in a direction perpendicular to the joint. This force being added to those which before entered into the discussion, viz., the weight of ASTD and the force P, will give a new direction and amount to the resultant R, which direction and amount are nevertheless determinable by the same methods of calculation. A new direction and amount of the force R for every position of S T brings with it a new form and direction of the line of resistance, and of the line of pressure. These are nevertheless determined, subject to the influence of these new elements of the calculation, precisely as before.

There is thus obtained the line of resistance and the line of pressure of the structure, on the supposition that the adhesive power of each of its joints is called into operation. Now on this hypothesis it is, as it was before, a necessary condition of the equilibrium, that the line of resistance shall pass actually through each joint of the structure, avoiding none; for if it avoid any joint, as for instance MN, fig. 2, then the resultant of all the forces acting upon the mass ACMN of the structure, there being included amongst these the whole amount of the adhesion of the surface MN, will act through some point *n* without that joint, and no force what-

ever opposing itself to the tendency of this resultant pressure to turn the mass over upon the point N, it will of necessity revolve upon that point.

The whole reasoning, this new element of adhesion being admitted into the discussion, becomes from this point, therefore, the same as before. Each semi-arch being supposed to require an opposite semi-arch to support it, so that neither would stand of itself, it follows that the line of resistance must somewhere in each semi-arch *meet* the intrados. Provided it does not also *cut* the extrados of the arch or of the pier, it will *touch* the intrados at the points where it thus meets it. The arch in this case will stand firmly. These points of contact of the line of resistance with the intrados are, as before, the points of rupture of the cemented arch: their position connects itself by a direct relation with the amount of the horizontal thrust, which amount is determined precisely as in the case of the uncemented arch. The case in which either semi-arch would stand without the support of an opposite semi-arch (which is the case of Mr. Brunel's experimental arch), is indicated by an evanescent or an impossible value of the horizontal thrust, resulting from the failure in this case of the hypothesis of a contact between the line of resistance and the intrados. It is evident that when the semi-arch will stand unsupported, no such contact takes place.

I have here shown by what process of calculation the adhesion of the cement of a structure may be included among the theoretical conditions of its equilibrium; but it has not been with the view of recommending the con-

sideration of this element in any practical question of construction. That structure (being of large dimensions) which would not stand without cement would assuredly be a perilous one.

A mass of masonry possessing few or none of those qualities which we understand under the term elasticity, the least appreciable alteration in the relative positions of its parts, such as not unfrequenly results from a settlement of its material, or an irregular sinking of its foundations, becomes a permanent disruption. No adhesive properties of the cement can control the disturbing forces called into operation whilst the arch is in the act of settling. In the case of an arch, such a disruption of the material is most likely to take place about the points of rupture. The adhesion of the cement is there then especially liable to be destroyed; and if this adhesion has been made an element in the calculation whence the proportions of the arch have resulted, so as to form a necessary condition to the equilibrium, then its destruction over these points of rupture will infallibly destroy the stability.[1]

That old principle of construction, then, which assigns to the structure such dimensions and proportions of its parts as would cause it to stand firmly were no cement used in its masonry, ought always to retain its authority with the judicious engineer.

[1] The destruction of the adhesive qualities of the cement over *any* point of the arch would *not* cause its overthrow, but only that over these points of rupture.

## CONDITIONS OF THE EQUILIBRIUM OF A STRUCTURE OF CONCRETE, OF COURSES OF STONES, OR OF BRICK-WORK.

Hitherto this discussion of the conditions of the equilibrium of a structure of masonry has assumed it to be capable of disruption only in the given directions of the joints by which the single stones which compose it are in contact with one another, or where the cement is interposed between them.

This hypothesis does not however include the whole question of its stability. The stones or bricks which compose such a structure may themselves be capable of disruption, and being placed in layers or courses as the stones or bricks of an edifice usually are, they may be in contact not only by their superior and inferior surfaces but by their sides; all their faces being united by cement and their joints breaking one another. A structure thus formed is evidently capable of disruption, through any given point, in an *infinite number* of different directions, and, which of these is the direction of its actual rupture is an element of the theory to be determined.

From whatever point of the surface of the structure the rupture may commence, its direction is subject to this condition, that it is that direction in which the resistance to rupture—as opposed to the particular forces which in each case tend to produce it—is the least; or, in other words, supposing the rupture to take place through any given point in the intrados or extrados

of the structure, then will the surface of rupture be that surface passing across the structure through this point, over which the various resistances opposing themselves to the rupture are the *least*. Thus, for instance, in fig. 18, if N be a point in the intrados of an arch about which rupture is about to take place, then will the direction of this rupture be such that the surface of rupture MN shall be that surface passing through N, on which the resistances opposing themselves to the rupture are, in their aggregate effect, the least. In a structure homogeneous as to the qualities of its materials and their distribution, this surface of least resistance to rupture is nearly a *plane* ;[1] for let M and N be points on the opposite side of the surface of rupture, then since the shortest line which can be drawn between N and M is the *straight* line between them, supposing the resistance to rupture to act uniformly over the surface of rupture, that resistance will be least between M and N, when the surface of rupture coincides with the straight line MN ; and the same being true for all other points similarly situated with M and N, it is evident that the surface of least resistance to rupture is a surface which may be generated by the motion of a straight line.

This would be *strictly* true were there any case of

[1] This remark does not extend to the case of the rupture by transverse strain of masses of greater compressibility than masonry, as for instance, masses of wood and iron in which the neutral axis lies, at the instant of rupture, *within* the surface of the mass, one side of it yielding by the compression, whilst the other yields by the extension of its material.

a structure throughout which the various forces opposing themselves to rupture were distributed with perfect uniformity. A structure of concrete would approach nearest to this case. In a cemented structure of the ordinary construction the surfaces of least resistance follow, for the most part, the joints of the bricks or stones, the adhesion of these to one another by their joints being less, where the interposed cement is mortar, than the adhesion of the individual parts of each brick or stone to one another. Subject to this condition the surface of least resistance still however approaches as nearly as possible to a plane surface when the materials of the structure are uniformly distributed.

Where the materials are not uniformly distributed, as in the arch, fig. 19, where the joints of the voussoirs have one direction, and those of the mass of masonry which it supports another, it is evident that the surface of least resistance to rupture, which follows the joint of the voussoir from N to M, will above the point M take a new direction ML, forming on the whole a line of broken inclination NML.

To simplify the discussion of these new elements of the theory, let there be supposed a uniform arrangement of the materials of the structure, and a uniform distribution of the forces resisting its rupture, as in the case of a structure of concrete, any deviations from this law which may occur in practice will be in favour of its stability, if we assume the unit of resistance to rupture at the least value which any part of the structure supplies.

Let any plane MN, fig. 18, be imagined to intersect the mass, inclined at an angle $\Theta$ to the horizon. Let the line of resistance be determined as in the case of an uncemented structure; if this line pass actually through the section MN there will be no tendency to rupture by the turning of either portion of the mass about the edge of that section. Suppose however that it does not, and that its direction is so far beyond these limits that the whole resistance to rupture on the plane MN is called into operation. This resistance may be estimated in terms of the unit of resistance, the inclination $\Theta$ of the plane of intersection, and the ordinate BO of the point N. The mass ABNM being imagined to be acted upon by this resistance, its weight, and the pressure P, let the resultant of these three forces, and its point of intersection with the plane MN be determined, and the plane being then conceived to take up an infinity of different positions by the variation of BO, let the *locus* of these intersections be found; let then P be assumed to have such a value as to give to this line of resistance a common point with the surface of the mass. This condition will serve to determine a relation between the force P, the ordinate BO of the point N, where the line of resistance meets the surface, and the inclination $\Theta$ of the intersecting plane. If the rupture of the structure be not about to take place at some point above N, it will evidently be upon the point of taking place at N, along a plane whose inclination is $\Theta$. But to every value of $\Theta$ corresponds a different value of the ordinate BO, and a different position of N subject to these con-

ditions ; so that at some point above N the rupture may be about to take place along a plane inclined at some other angle $\Theta$ to the horizon. Determine then the *minimum* value of the ordinate BO in respect to the variable $\Theta$ of which it is a function, or the least height at which any inclination of the plane of rupture can cause an intersection between the line of resistance and the surface of the structure, the corresponding point N will be the true point of rupture ; above that point the parts of the structure will not be separated by the pressures to which they are subjected, but at that point they will be upon the point of separating. This point determined, the discussion of the further conditions of the equilibrium is precisely the same as before. If the line of resistance thus determined cut the surface of the structure at N, actual rupture will take place ; if it merely touch, as in the case of the arch, there will be no rupture, but a state preserved which is always one bordering on rupture. The determination of this point has however supposed a knowledge of the circumstances under which rupture may take place along the plane MN of rupture, and of the unit of force opposing itself to rupture on that plane. The circumstances of this rupture are of two kinds. It may take place along the plane MN by the *turning* of the mass ABNM upon the point N, the forces opposing themselves to which rupture are the direct adhesions of the parts of the mass along that plane and perpendicular to it, and this is the case above supposed ; or it may take place by the sliding of the mass ABNM′ along the plane of rupture NM′, to which

rupture are opposed, 1st, those *cohesive* forces of the mass which oppose themselves to the sliding of any two portions of it asunder, and which are independent of the insistent pressures; and, 2ndly, the forces of friction which oppose themselves to the motion of any two portions of the mass upon one another when separated, and which are dependent upon the insistent pressures.

Now the forces thus entering into the discussion of both these cases admit of determination in terms of known units of each, and may therefore be made to enter with precision into the formulæ of equilibrium.

In the consideration of the forces which oppose themselves to rupture in the first case, where they act in directions perpendicular to the plane MN, there is a difficulty in making the proper estimate of those *compressible* and *elastic* properties which are common in different degrees to all bodies, but which may, perhaps, be considered to be called less actively into operation in a mass of masonry than in any other. We may at any rate assign limits within which their influence on the stability of a structure must be confined. Let us first suppose the adhesion of the surfaces of contact in MN not to be at all affected by the elastic properties of the material; the resultant of all the elementary adhesions of the parts of that plane will then pass through its centre of gravity, or, if it be a rectangle, as is commonly the case, the resultant will pass through the centre of that rectangle. Next let us suppose the surfaces of contact in MN to be united by a stratum of some perfectly elastic material, a lamina of caoutchouc

for instance; the extension of this elastic material will then be different at different distances from N, varying, when the surfaces turn upon N, directly at these distances; and in the same proportion will vary the forces by which at different points the two surfaces are held together. Now from this it may readily be calculated by the principles of the integral calculus, that the point through which the resultant of all these elementary elastic forces, holding the two surfaces together, passes, is no longer situated in the centre of the rectangle, but at a distance of $\frac{1}{6}$th the distance MN beyond that centre. So that supposing the aggregate adhesion to be in this case the same as before, its efficiency to maintain the equilibrium will be increased by an increased leverage of this amount. Now between these two limits of perfect elasticity and compressibility, and perfect rigidity of the material opposing its adherence to rupture in MN, the true case of the adherence of a mass of masonry may be considered to lie. The stability will certainly not be less than that given by the first hypothesis, and not greater than that given by the second. It is because the first hypothesis thus assigns the lowest estimate to the influence on the stability of adhesive properties in the cement, that I have assumed it as the basis of the method explained in my last paper (page 29), for determining the conditions of the equilibrium of a cemented structure of single stones.

To determine the conditions of rupture in the second case where the structure yields by sliding upon the plane NM′, assume any inclination $\Theta$ to the plane of rupture

NM′, fig. 18, determine the direction of the resultant of the pressures upon ABNM′, including amongst these pressures the pressure P, the weight of the mass AB NM′, and the forces of adhesion which oppose themselves to the sliding of the surfaces of contact in NM′ upon one another independently of the friction. If the inclination of this resultant to the perpendicular to NM′ be greater than the limiting angle of resistance, rupture will take place. Suppose it to equal this angle so that rupture may be upon the point of taking place. A relation will thus be determined between $\Theta$ and BO the ordinate of the point N, so that for that position of N rupture would be upon the point of taking place along a plane whose inclination is $\Theta$. The minimum value of BO, considered as a function of $\Theta$ which satisfies this condition, determines the point of rupture. At that point two portions of the mass will slide upon one another, and at no point above it.

The two cases will give different geometrical forms to the lines of resistance and different positions of the point of rupture N. The true point of rupture of the two thus determined, will be the highest.

In the great majority of structures, however, we may be assured that the first case is the only one which it is necessary to consider, rupture by the sliding asunder of two portions of the mass across its solid material rarely occurring.

## THE MATHEMATICAL THEORY OF THE EQUILIBRIUM OF A STRUCTURE.

In order that the principles laid down in the preceding papers may become intelligible when translated into the language of analysis under their more general forms, it will be well, first of all, to present to the reader some of their simpler and more elementary applications.

I propose then, *first*, to consider the equilibrium of the PIER; *secondly*, that of the loaded CIRCULAR ARCH with equal voussoirs; *thirdly*, I propose to discuss those *general methods of analysis* by which the lines of resistance and pressure may be determined in ANY STRUCTURE subject to *any* given circumstances of pressure, and whence may be deduced the actual conditions of the equilibrium of that structure. Of this last determination, all the others are, of course, particular cases, and the order of investigation which I here propose to myself is the opposite of that which I followed in the original consideration of the question.

### THE PIER.

Let ABCD, fig. 20, represent an upright pier of uniform dimensions to any point in the summit AB, of which is applied a given pressure P. To determine the line of resistance in this pier let MN be conceived to be any horizontal section of it. The forces impressed upon the portion AMNB of the pier are then, its weight

W, which may be conceived to be collected in the direction of its axis and applied at S, and the pressure P on the summit of the pier. Let R be the resultant of W and P, then is L, where the direction of R intersects MN, a point in the line of resistance. Let $ES=x$, $SL=y$, a relation established between $x$ and $y$ will determine the position of the point L for every position of the intersecting plane, and will be the equation to the line of resistance.

Let AB, the width of the pier, be represented by 2K, and the weight of a cubical unit of the pier by $\mu$. Let the distance EQ of the point of application of the force P from the axis of the pier be $k$, and the inclination of PQ to the vertical $\Phi$; also let the inclination of R to the vertical be $\theta$.

Then the forces P, W, R, being resolved vertically and horizontally become

$$\left.\begin{array}{lll} P\cos.\,\Phi, & W, & R\cos.\,\theta \\ P\sin.\,\Phi, & & R\sin.\,\Phi \end{array}\right\}$$

therefore, by the general conditions of the equilibrium of forces in the same plane, observing that the co-ordinates of the points of application of P, W, and R, are respectively

$$\begin{array}{c} -k,\ 0 \\ 0,\ +x \\ +y,\ +x \end{array}$$

$$\left.\begin{array}{l} P\cos.\,\Phi+W-R\cos.\,\theta=0 \\ P\sin.\,\Phi-R\sin.\,\theta=0 \\ -k\,P\cos.\,\Phi-Ry\cos.\,\theta+Rx\sin.\,\theta=0 \end{array}\right\} \quad \ldots\ldots\ (1)$$

Whence, observing that $W=2K\mu x$, we obtain by the elimination of R and $\theta$ in the last equation

$$-k\,P\cos.\,\Phi-y\,(P\cos.\,\Phi+2K\mu x)+P\,x\sin.\,\Phi=o$$

$$\therefore\ y=P.\frac{x\sin.\,\Phi-k\cos.\,\Phi}{2K\mu x+P\cos.\,\Phi}\quad\ldots\ldots\ldots\quad(2)$$

which is the required equation to the line of resistance.

### SECOND METHOD OF DETERMINING THE LINE OF RESISTANCE.

To persons unaccustomed to the application of the general equations of equilibrium the following method of determining the line of resistance in a pier will be more intelligible.

Take OU=SE (fig. 21) to represent the weight W of the mass ABNM. Produce PQ to V, and take OV to represent the force P on the same scale on which OU represents W. Complete the parallelogram OUYV, then OY will represent the resultant R in magnitude and direction, and L will be a point in the line of resistance.

Draw XY parallel to SL. Therefore by similar triangles,

$$\frac{SL}{OS}=\frac{XY}{OX}=\frac{UY\sin.\,XUY}{OU+UY\cos.\,XUY}$$

$$=\frac{P\sin.\,XUY}{W+P\cos.\,XUY}\quad\ldots\ldots\ldots\quad(1)$$

Also adopting the same notation as before

$$SL=y,\ OS=ES-EO=x-k\cot.\,\Phi,\ XUY=\Phi,\ W=2K\mu x$$

$$\therefore \frac{y}{x-k\cot.\Phi} = \frac{P\sin.\Phi}{2K\mu x+P\cos.\Phi}$$

$$\therefore y=P\,\frac{x\sin.\Phi-k\cos.\Phi}{2K\mu x+P\cos.\Phi} \quad \ldots\ldots\ldots (2)$$

If P′ represent the solidity of a mass of the same material as the pier whose weight shall equal P, then

$$P=P'\mu$$

$$\therefore y=P'.\,\frac{x\sin.\Phi-k\cos.\Phi}{2Kx+P'\cos.\Phi} \quad \ldots\ldots\ldots (3)$$

Now this equation is that to a rectangular hyperbola, as will be apparent from the following transformations.

$$y\{2Kx+P'\cos.\Phi\}=P'x\sin.\Phi-P'k\cos.\Phi$$

$$\therefore y\left\{x+\frac{P'\cos.\Phi}{2K}\right\}-\frac{P'\sin.\Phi}{2K}x=-\frac{P'k\cos.\Phi}{2K}$$

Subtracting $\frac{P'^2\sin.\Phi\cos.\Phi}{4K^2}$ from both sides

$$y\left\{x+\frac{P'\cos.\Phi}{2K}\right\}-\frac{P'\sin.\Phi}{2K}\left\{x+\frac{P'\cos.\Phi}{2K}\right\}$$

$$=-\frac{P'\cos.\Phi}{2K}\left\{\frac{P'\sin.\Phi}{2K}+k\right\}$$

$$\therefore \left\{\frac{P'\sin.\Phi}{2K}-y\right\}\left\{x+\frac{P'\cos.\Phi}{2K}\right\}=\frac{P'\cos.\Phi}{2K}$$

$$\left\{\frac{P'\sin.\Phi}{2K}+k\right\}$$

Let EG, fig. 22, be taken $=\frac{P'\sin.\Phi}{2K}$

GH . . . . . $=\frac{P'\cos.\Phi}{2K}$

ES$=x$ SL$=y$

$$\therefore \text{IL} = \frac{\text{P}' \sin.\Phi}{2\text{K}} - y$$

$$\text{IH} = \frac{\text{P}' \cos.\Phi}{2\text{K}} + x$$

$$\therefore \text{IL} \cdot \text{IH} = \left\{\frac{\text{P}' \sin.\Phi}{2\text{K}} - y\right\} \left\{\frac{\text{P}' \cos.\Phi}{2\text{K}} + x\right\}$$

$$= \frac{\text{P}' \cos.\Phi}{2\text{K}} \left\{\frac{\text{P}' \sin.\Phi}{2\text{K}} + k\right\}$$

$= a$ constant for all the joints of the same pier subjected to the same insistent pressures.

Since then the rectangle of the co-ordinates of the line of resistance in respect to the axis HX is for every point of that line the same, it follows, by a well known property, that its geometrical form is that of a rectangular hyperbola, of which HX is the asymptote.

The line of resistance therefore continually *approaches* HX, but never meets it. If then HX lies, as shown in the figure, actually *within* the mass of the pier, or if EG be less than EB or $\frac{\text{P}' \sin.\Phi}{2\text{K}}$ less than K, or $\text{P}' \sin.\Phi$ than $2\text{K}^2$, then the line of resistance will no where cut the extrados of the pier, and the structure will retain its stability under the insistent pressure P, however high it may be built.

If HK lie *without* the mass of the pier, or $\text{P}' \sin.\Phi$ be not less than $2\text{K}^2$, then the line of resistance will somewhere cut the intrados of the pier, provided it be built of a sufficient height, and this height will evidently be the greatest to which the pier can be built without being overthrown ; it is evidently equal to that value of $x$

which is determined in the equation 3 to the line of resistance by assuming $y$ equal to K, or to one-half the width of the pier.

This substitution being made, and the corresponding value of $x$, or the greatest possible height of the pier, being represented by H, we have

$$2\,K^2\,H + P'\,K\cos.\,\Phi = P'\,H\sin.\,\Phi - P'\,k\cos.\,\Phi$$

$$\therefore H = \frac{(K+k)\,P'\cos.\,\Phi}{P'\sin.\,\Phi - 2\,K^2} \quad \ldots\ldots\ldots \quad (4)$$

It will be observed that P sin. $\Phi$ and P cos. $\Phi$ are the resolved parts, horizontally and vertically, of the insistent pressure P, and that P′ sin. $\Phi$ and P′ cos. $\Phi$ are the volumes of the same material as the pier, whose weights would be equal to these resolved pressures; moreover, that if these volumes be supposed to be those of masses of the same section with the pier, then their *heights* will be represented by $\frac{P'\sin.\,\Phi}{2\,K}$ and $\frac{P'\cos.\,\Phi}{2\,K}$.

RULE TO DETERMINE, BY CONSTRUCTION, THE DIRECTION AND AMOUNT OF THE RESULTANT PRESSURE UPON ANY JOINT OF A PIER.

Let MN, fig. 21, be any joint of the pier. And PQ the resultant of the pressures which it bears. Produce PQ to V intersecting the axis EX in O. Take OU = ES, and OV equal to the height of a mass, which being of the same section and material as the pier, shall have a weight equal to the pressure P. Complete the parallelogram OUYV, and draw its diagonal OY, then

will this line be in the direction of the resultant of the pressures upon the joint MN; its length will, moreover, equal the height of a mass of the same section and material as the pier, the weight of which mass would just equal the amount of this resultant pressure.

It is evident, since the resultant pressures on all the joints pass through the point O, and have their directions between the lines OV and OX, that they are all inclined to the vertical at less angles than the force P; so that if the direction of PQ be not without the limiting angle of resistance, then will the directions of the resultants of the pressures on none of the joints be without that angle, so that none of the stones of the pier will be made to slip upon one another; it can only therefore fall by being made to turn on the edges of some of its subjacent stones. The truth of the above rule is evident from the theory given in page 43.

RULE TO DETERMINE, BY CONSTRUCTION, WHETHER A PIER BUILT UP OF SINGLE STONES, WITHOUT CEMENT, WILL STAND UNDER THE PRESSURES TO WHICH IT IS SUBJECTED, TO WHATEVER HEIGHT IT MAY BE BUILT.[1]

Resolve the pressures sustained by the pier[2] vertically

[1] This and the following rules are extracted from the author's work entitled "Illustrations of Mechanics," page 209.

[2] It will be observed that if in fig. 21 the distance EQ or $k$ exceed EA, the pressure P will be applied on the face or intrados AM of the pier; this case is therefore fully included in the results of the preceding discussion.

and horizontally. Calculate the height of a mass, which being of the same substance and the same section as the pier, shall have a weight equal to the sum of those pressures which are thus resolved horizontally. Let AS, fig. 23, be this height. From the centre E of the pier measure EG equal to AS, and draw the vertical line CGX; this line will be the asymptote to the line of resistance, which has been shown to be a hyperbola. If GX lie within the mass of the structure, or if EG be less than EB, or less than half the width of the pier, then the structure will stand, to however great a height it may be built. If EG be greater than EB, so that GX lies *without* the structure, then the pier cannot be built above a certain height, so as to stand under the insistent pressures.

### RULE TO DETERMINE BY CONSTRUCTION THE GREATEST HEIGHT TO WHICH A PIER CAN BE BUILT SO AS TO SUSTAIN GIVEN INSISTENT PRESSURES.

Let P, fig. 24, represent the point where the resultant of the insistent pressures intersects the summit of the pier, and let AS and AT represent the heights of masses which, being of the same section and the same material as the pier, shall equal in weight respectively, the sum of the insistent pressures when resolved horizontally, and their sum when resolved vertically.

Take EG equal to AS, E being the centre of the pier, and let the point G be beyond B, so that the pier will (by the last rule) be overthrown, if raised above a certain

height to be determined; join G U, and through the point P draw P Z parallel to G U, then will B Z be the greatest height to which the pier can be raised, or if it be carried higher, then is this the point to which an inclined buttress should be built to support it.[1]

TO DETERMINE THE LINE OF RESISTANCE IN AN ARCH WHOSE INTRADOS IS A CIRCLE, WHOSE EXTRADOS IS OF ANY GIVEN GEOMETRICAL FORM, AND WHICH SUPPORTS ANY GIVEN LOAD.

LET ADBF, fig. 25, represent any portion of such an arch, P a pressure applied at its extreme voussoir, and X and Y the horizontal and vertical components of any pressure borne upon its extrados, or of the *resultant* of any number of such pressures; let moreover the co-ordinates, from the centre C, of the point of application of this pressure, or this *resultant* pressure, be $x$ and $y$.

Let the horizontal force P be applied in A D at a vertical distance $p$ from C; also let C T represent any plane which, passing through C, intersects the arch in a direction parallel to the joints of its voussoirs.

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[1] For by similar triangles

$$\frac{BZ}{BP} = \frac{BU}{BK} = \frac{AT}{AS - EB},$$

$$\therefore \frac{BZ}{(K+k)} = \frac{\left(\frac{P' \cos. \Phi}{2K}\right)}{\left(\frac{P' \sin. \Phi}{2K} - K\right)};$$

$$\therefore BZ = \frac{(K+k)\, P' \cos. \Phi}{P' \sin. \Phi - 2K^2} = H' \text{ by equation (4).}$$

Let this plane be intersected by the *resultant* of the pressures applied to the mass ASTD in R. These pressures are the weight of the mass ASTD, the load X and Y, and the pressure P. Now if pressures equal and parallel to these, but in opposite directions, were applied at R, they would of themselves support the mass, and the whole of the subjacent mass TSB might be removed without affecting the equilibrium. Imagine this to be done; call M the weight of the mass ASTD, and $h$ the horizontal distance of its centre of gravity from C, and let CR be represented by $\rho$, and the angle ECS by $\theta$, then the perpendicular distances from C of the pressures M+Y and P−X, imagined to be applied to R, are $\rho \sin. \theta$ and $\rho \cos. \theta$; therefore by the condition of the equality of moments

$$(M+Y)\rho \sin.\theta+(P-X)\rho \cos.\theta=Mh+Yx-Xy+Pp. \quad (5)$$

$$\therefore \rho=\frac{Mh+Yx-Xy+Pp}{(M+Y) \sin. \theta+(P-X) \cos. \theta},$$

which is the equation to the line of resistance.

M and $h$ are given functions of $\theta$, as also are X and Y, if the pressure of the load extend *continuously* over the surface of the extrados from D to T.

It remains from this equation to determine the pressure P, being that supplied by the opposite semi-arch. As the simplest case, let all the voussoirs of the arch be of the same depth, and let the inclination ECP of the first joint of the semi-arch to the vertical be represented by $\Theta$, and the radii of the intrados and extrados by R and $r$. Then by the known principles of statics,

$$M\,h=\int_r^R\int_\Theta^\theta r^2 \sin.\,\theta\,d\theta\,dr=-\tfrac{1}{3}\,(R^3-r^3)\,(\cos.\,\theta-\cos.\,\Theta),\text{ also } M=\tfrac{1}{2}\,(R^2-r^2)\,(\theta-\Theta)$$

$$\therefore \rho\left\{\tfrac{1}{2}\,(R^2-r^2)\,(\theta-\Theta)\sin.\,\theta+Y\sin.\,\theta-X\cos.\,\theta+P\cos.\,\theta\right\}=\tfrac{1}{3}\,(R^3-r^3)\,(\cos.\,\Theta-\cos.\,\theta)+Y\,x-X\,y+P\,p \quad (6)$$

Now at the points of rupture the line of resistance *meets* the intrados, so that there $\rho=r$; if then $\Psi$ be the corresponding value of $\theta$,

$$r\left\{\tfrac{1}{2}\,(R^2-r^2)\,(\Psi-\Theta)\sin.\,\Psi+Y\sin.\,\Psi-X\cos.\,\Psi+P\cos.\,\Psi\right\}=\tfrac{1}{3}\,(R^3-r^3)\,(\cos.\,\Theta-\cos.\,\Psi)+Y\,x-X\,y+P\,p\ . \quad (7)$$

Also at the points of rupture the line of resistance *touches* the intrados, so that there $\frac{d\rho}{d\theta}=\frac{dr}{d\theta}=0$; assuming then, to simplify the results, that the presure of the load is wholly in a vertical direction, so that $X=0$, and that it is collected over a single point of the extrados, so that $\frac{dY}{d\theta}=0$, and differentiating equation (6), and assuming $\frac{d\rho}{d\theta}=0$ when $\theta=\Psi$ and $\rho=r$, we obtain

$$r\left\{\tfrac{1}{2}\,(R^2-r^2)\,(\Psi-\Theta)\cos.\,\Psi+\tfrac{1}{2}\,(R^2-r^2)\sin.\,\Psi+Y\cos.\,\Psi-P\sin.\,\Psi\right\}=\tfrac{1}{3}\,(R^3-r^3)\sin.\,\Psi;$$

hence, assuming $R=r\,(1+\alpha)$,

$$\left\{\frac{6\,P}{r^2}+\alpha^2\,(2\,\alpha+3)\right\}\tan.\,\Psi=\left\{\frac{6\,Y}{r^2}-3\alpha\,(\alpha+2)\,\Theta\right\}+3\,\alpha\,(\alpha+2)\,\Psi \quad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

Eliminating $(\Psi-\Theta)$ between equations (7) and (8), we have

$$\left\{\frac{P}{r^2}+\alpha^2\,(\tfrac{1}{3}\,\alpha+\tfrac{1}{2})\right\}\text{sec.}^2\Psi-\left\{\frac{Y\,x+P\,p}{r^3}+\alpha\,(\tfrac{1}{3}\,\alpha^2+\alpha+1)\cos.\,\Theta\right\}\text{sec.}\,\Psi=-\alpha\,(\tfrac{1}{2}\alpha+1) \quad \ldots\ldots\ldots (9)$$

Eliminating P between equations (7) and (8), and reducing

$$\frac{Y^*}{r^2}\left\{\frac{p\cos.\Psi+x\sin.\Psi}{r}-1\right\}=(\tfrac{1}{2}\alpha^2+\alpha)\,(1-\frac{p}{r}\cos.\Psi)\,(\Psi-\Theta)+\frac{p}{r}(\tfrac{1}{2}\alpha^2+\tfrac{1}{3}\alpha^3)\sin.\Psi-$$

$$\left\{(\alpha+\alpha^2+\tfrac{1}{3}\alpha^3)\cos.\Theta-(\tfrac{1}{2}\alpha^2+\alpha)\cos.\Psi\right\}\sin.\Psi \quad \ldots\ldots\ldots\ldots\ldots\ldots (10)$$

Let A P, fig. 25, $=\lambda r$ $\therefore$ $\frac{p}{r}=(1+\lambda)\cos.\Theta$. Substituting this value of $\frac{p}{r}$,

$$\frac{Y}{r^2}\left\{\frac{x}{r}\sin.\Psi+(1+\lambda)\cos.\Theta\cos.\Psi-1\right\}=(\tfrac{1}{2}\alpha^2+\alpha)\left\{\{1-(1+\lambda)\cos.\Theta\cos.\Psi\}(\Psi-\Theta)+(\cos.\Psi-\cos.\Theta)\sin.\Psi\right\}$$

$$+\lambda(\tfrac{1}{2}\alpha p+\tfrac{1}{3}\alpha^3)\sin.\Psi\cos.\Theta \quad \ldots\ldots\ldots\ldots\ldots\ldots (11)$$

If the arch be a continuous segment the joint AD is vertically above the centre, and CD coinciding with CE, $\Theta=o$; if it be

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* This equation might have been obtained by differentiating equation (6) in respect to P and $\theta$, and assuming $\frac{d\,P}{d\,\theta}=0$ when $r$ and $\Psi$ are substituted for $\rho$ and $\theta$; for if that equation be represented by $u=0$, $u$ being a function of P, $\rho$ and $\theta$, $\frac{d\,u}{d\,P}\frac{d\,P}{d\,\theta}+\frac{d\,u}{d\,\theta}=0$, and $\frac{d\,u}{d\,\rho}\frac{d\,\rho}{d\,\theta}+\frac{d\,u}{d\,\theta}=0$. The same result $\frac{d\,u}{d\,\theta}=0$ is therefore obtained whether we assume $\frac{d\,P}{d\,\theta}=0$, or $\frac{d\,\rho}{d\,\theta}=0$, which last supposition is that made in equation (8), whence equation (10) has resulted. The hypotheses $\frac{d\,P}{d\,\theta}=0$, $\rho=r$, determine the minimum of the pressures P, which being applied to the key-stone will prevent the semi-arch from turning on any of the successive joints of its voussoirs.

a broken segment, as in the Gothic arch, $\Theta$ has a given value determined by the character of the arch. In the pure or equilateral Gothic arch $\Theta=30°$. Assuming $\Theta=0$, and reducing

$$\frac{Y}{r^2}\left\{\frac{x}{r}-\left(\text{tan.}\frac{\Psi}{2}-\lambda\ \text{cot.}\ \Psi\right)\right\}=(\tfrac{1}{2}\alpha^2+\alpha)\left\{\left(\text{tan.}\frac{\Psi}{2}-\lambda\ \text{cot.}\ \Psi\right)\Psi-\text{vers.}\ \Psi\right\}+\lambda\ (\tfrac{1}{2}\alpha^2+\tfrac{1}{3}\alpha^3)\quad\ldots\ldots\ (12)$$

It may easily be shown that as $\Psi$ increases in this equation Y increases and conversely, so that as the load is increased the points of rupture descend. When $Y=0$, or there is no load upon the extrados,

$$\left(\text{tan.}\frac{\Psi}{2}-\lambda\ \text{cot.}\ \Psi\right)\Psi-\text{vers.}\ \Psi+\tfrac{1}{3}\lambda\alpha\ \frac{3+2\alpha}{2+\alpha}=0\quad\ldots\ldots\ldots\ldots\ldots\ldots\ (13)$$

When $x=0$, or the load is placed on the crown of the arch,

$$\frac{Y}{r^2}=\frac{(\frac{1}{2}\alpha^2+\alpha)\ \text{vers.}\ \Psi-\lambda\ (\frac{1}{2}\alpha^2+\frac{1}{3}\alpha^3)}{\text{tan.}\frac{\Psi}{2}-\lambda\ \text{cot.}\ \Psi}-(\tfrac{1}{2}\alpha^2+\alpha)\ \Psi\quad\ldots\ldots\ldots\ldots\ (14)$$

When $\frac{x}{r}-\left(\text{tan.}\frac{\Psi}{2}-\lambda\ \text{cot.}\ \Psi\right)=0$, $\frac{Y}{r^2}$ becomes infinite; an infinite load is therefore required to give that value to the angle of rupture which is determined by this equation. Solved in respect to tan. $\frac{\Psi}{2}$, it gives

$$\text{tan.}\frac{\Psi}{2}=\frac{\left(\frac{x}{r}\right)+\sqrt{\left(\frac{x}{r}\right)^2+\lambda\ (2+\lambda)}}{2+\lambda}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ (15)$$

No loading placed upon the arch can cause the angle of rupture to exceed that determined by this equation.

Let us now consider the case, in which the pressure of the load is distributed over different points of the extrados; let it be supposed that this pressure is wholly vertical, and such that any portion F T of the extrados (fig. 29) sustains the weight of a mass G F T V immediately superincumbent to it, and bounded by the straight line G V inclined to the horizon at the angle $\iota$; let moreover the weight of each cubical unit of the load be equal to that of the same unit of the material of the arch, multiplied by the constant factor $\mu$; then representing A D by R$\beta$, we have

$$\text{Y}=\text{mass G F T V}=\mu\,\text{R}^2\int_{\Theta}^{\theta}\left\{1+\beta-\text{sec. }\iota\cos.(\theta-\iota)\right\}\cos.\theta\,d\theta=\mu\,\text{R}^2\Big\{(1+\beta)(\sin.\theta-\sin.\Theta)-\tfrac{1}{4}\text{ sec. }\iota\{\sin.(2\theta-\iota)$$
$$-\sin.(2\Theta-\iota)\}-\tfrac{1}{2}(\theta-\Theta)\Big\}\quad\ldots\ldots\ldots\ldots\ldots\ldots\quad(16)$$

$$\text{Y}x=\text{momentum of G F T V}=\mu\,\text{R}^3\int_{\Theta}^{\theta}\left\{(1+\beta)-\text{sec. }\iota\cos.(\theta-\iota)\right\}\sin.\theta\cos.\theta\,d\theta=\mu\,\text{R}^3\Big\{\tfrac{1}{2}(1+\beta)(\cos.^2\Theta-\cos.^2\theta)$$
$$-\tfrac{1}{3}(\cos.^3\Theta-\cos.^3\theta)-\tfrac{1}{3}\tan.\iota\,(\sin.^3\theta-\sin.^3\Theta)\Big\}\quad\ldots\ldots\ldots\quad(17)$$

As the simplest case let us suppose D V horizontal, and the crown of the arch to be at A, so that $\iota=0$, and $\Theta=0$. Substituting the values of Y and Y$x$ which result from these suppositions, in equation (7), solving that equation in respect to $\frac{\text{P}}{r^2}$ and reducing, we have

$$\frac{\text{P}}{r^2}=\frac{(1-\alpha)(1+\alpha)^2(1+\beta)\sin.^2\Psi+\frac{1}{6}(1+\alpha)^2(1-2\alpha)\cos.^3\Psi+(\frac{1}{2}\alpha^2+\frac{1}{3}\alpha^3-\frac{1}{2})\cos.\Psi-\frac{1}{2}\Psi\sin.\Psi+\frac{1}{3}}{1+\lambda-\cos.\Psi}\quad\ldots(18)$$

Assuming $\frac{d\text{P}}{d\Psi}=0$ (see note, page 52), and $\lambda=\alpha$, and reducing

$$\tfrac{2}{3}(1-2\alpha)(1+\alpha)^2\cos.^3\Psi-\left\{(1-\alpha)(1+\beta)+(1+\alpha)(1-2\alpha)\right\}\cos.^2\Psi+\left\{\frac{1}{(1+\alpha)^2}+2(1-\alpha^2)(1+\beta)\right\}\cos.\Psi$$

$$+\frac{1}{(1+\alpha)^2}\left\{1-(1+\alpha)\cos.\Psi\right\}\frac{\Psi}{\sin.\Psi}-(1-\alpha)\beta-\frac{2}{3(1+\alpha)^2}-\frac{1+\frac{2}{3}\alpha^3}{1+\alpha}=0 \quad \ldots\ldots\ldots \quad (19)$$

In the case in which the line of resistance passes through the bottom of the key-stone so that $\lambda=0$, equation 18 becomes

$$\frac{P}{r^2}=\tfrac{1}{2}(1+\alpha)^2(1+\beta)(1-\alpha)(1+\cos.\Psi)-\tfrac{1}{6}(1+\alpha)^2(1-2\alpha)(1+\cos.\Psi)\cos.\Psi-\tfrac{1}{2}\Psi\cot.\tfrac{1}{2}\Psi+\tfrac{1}{3}=0 \quad \ldots \quad (19)$$

whence, assuming $\frac{dP}{d\Psi}=0$, we have

$$\tfrac{2}{3}(1+\alpha)^2(1-2\alpha)\cos.^2\Psi+(1+\alpha)^2\left\{\tfrac{5}{3}\alpha-(1-\alpha)\beta-\tfrac{4}{3}\right\}\cos.\Psi+\frac{\Psi}{\sin.\Psi}+1=0 \quad \ldots\ldots\ldots \quad (20)$$

Let us next take a case of *oblique* pressure on the extrados, and let us suppose it to be the pressure of *water*, whose surface stands at a height $\beta$ R above the summit of the key-stone. The pressure of this water being perpendicular to the extrados will every where have its direction through the centre C, so that its moment about that point will vanish, and $Yx-Xy=0$; moreover, by the principles of hydrostatics, the vertical component Y of the pressure of the water, superincumbent to the portion AT of the extrados, will equal the weight of that mass of water, and will be represented by the formula (16), if we assume $\iota=0$. The horizontal component X of the pressure of this mass of water is represented by the formula

$$X=\mu R^2\int_{\Theta}^{\theta}\left\{1+\beta-\cos.\theta\right\}\sin.\theta\,d\theta=\mu(1+\alpha)^2r^2\left\{(1+\beta)(\cos.\Theta-\cos.\theta)-\tfrac{1}{2}(\cos.^2\Theta-\cos.^2\theta)\right\} \quad \ldots \quad (21)$$

Assuming then $\Theta=0$, we have $\frac{Y}{r^2}=\mu(1+\alpha)^2\left\{(1+\beta)\sin.\Psi-\frac{1}{4}\sin.2\Psi-\frac{1}{2}\Psi\right\}$

$$\text{and } \frac{X}{r^2} = \mu\,(1+\alpha)^2\left\{(1+\beta)\text{ vers. }\Psi - \tfrac{1}{2}\text{ sin. }^2\Psi\right\}$$

$$\therefore \frac{Y}{r^2}\text{ sin. }\Psi - \frac{X}{r^2}\text{ cos. }\Psi = \mu\,(1+\alpha)^2\left\{(1+\beta)\text{ vers. }\Psi - \tfrac{1}{2}\,\Psi\text{ sin. }\Psi\right\} \quad \ldots\ldots\ldots\ldots \quad (22)$$

Substituting this value in equation (7), making $Yx - Xy = 0$, solving that equation in respect to $\frac{P}{r^2}$ and making $\frac{p}{r} = 1+\lambda$ we have

$$\frac{P}{r^2} = \frac{\{\frac{1}{2}\alpha^2+\alpha-\frac{1}{2}\mu\,(1+\alpha)^2\}\Psi\text{ sin. }\Psi - \{\alpha+\alpha^2+\frac{1}{3}\alpha^3-\mu\,(1+\alpha)^2\,(1+\beta)\}\text{vers. }\Psi}{\lambda+\text{vers. }\Psi} \quad \ldots\ldots\ldots \quad (23)$$

If, instead of supposing the pressure of the water to be borne by the extrados, we suppose it to take effect upon the intrados tending to *blow up* the arch, and if $\beta$ represent the height of the water above the crown of the intrados, we shall obtain precisely the same expressions for X and Y as before, except that $r$ must be substituted for $(1+\alpha)\,r$, and that both X and Y must be taken *negatively;* in this case therefore $\frac{Y}{r^2}\text{ sin. }\Psi - \frac{X}{r^2}\text{ cos. }\Psi = -\mu\left\{(1+\beta)\text{ vers. }\Psi - \frac{1}{2}\Psi\text{ sin. }\Psi\right\}$, whence, by substitution in equation (7), and reduction

$$\frac{P}{r^2} = \frac{(\frac{1}{2}\alpha^2+\alpha+\frac{1}{2}\mu)\,\Psi\text{ sin. }\Psi - \left\{\alpha+\alpha^2+\frac{1}{3}\alpha^3+\mu\,(1+\beta)\right\}\text{vers. }\Psi}{\lambda+\text{vers. }\Psi} \quad \ldots\ldots\ldots\ldots \quad (24)$$

Now, by note, page 52, $\frac{d\left(\frac{P}{r^2}\right)}{d\Psi} = 0$, differentiating equations (22) and (23) therefore, and reducing, we have

$$\Psi\left\{\text{tan. }\frac{\Psi}{2} - \lambda\text{ cot. }\Psi\right\} + \text{cos. }\Psi + A\lambda - 1 = 0 \quad \ldots\ldots\ldots\ldots\ldots \quad (25)$$

which equation applies to both the cases of the pressure of a fluid upon an arch with equal voussoirs; that in which its pressure is borne by the extrados, and that in which it is borne by the intrados, the constant A representing in the first case the quantity $\frac{\frac{1}{2}\alpha^2+\frac{1}{3}\alpha^3-\mu\,(\frac{1}{2}+\beta)\,(1+\alpha)^2}{\frac{1}{2}\alpha^2+\alpha-\frac{1}{2}\mu\,(1+\alpha)^2}$ and in the second case $\frac{\frac{1}{2}\alpha^2+\frac{1}{3}\alpha^3+\mu\,(\frac{1}{2}+\beta)}{\frac{1}{2}\alpha^2+\alpha+\frac{1}{2}\mu}$. If the line of resistance pass through the summit of the key-stone, $\lambda$ must be taken$=\alpha$. If it pass along the inferior edge of the key-stone, $\lambda=0$. In this second case, $\tan.\frac{\Psi}{2}\left\{\Psi-\sin.\Psi\right\}=0\;\therefore\;\Psi=0$, so that the point of rupture is at the crown of the arch. For this value of $\Psi$, (23) and (24) become vanishing fractions, whose values are determined by known methods of the different calculus to be,

When the pressure is on the extrados $\frac{P}{r^2}=\alpha-\frac{1}{3}\alpha^3+\beta\mu\,(1+\alpha)^2$ . . . . . (26)

When the pressure is on the intrados $\frac{P}{r^2}=\alpha-\frac{1}{3}\alpha^3-\beta\mu$ . . . . . (27)

The equations given above completely determine the value of P, subject to the first of the two conditions stated in page 13, viz., that the line of resistance passing through a given point in the key-stone determined by a given value of $\lambda$, shall have a point of geometrical *contact* with the intrados. It remains now to determine it subject to the second condition, viz., that its point of application P on the key-stone shall be such as to give it the least value which it can receive subject to the first condition. It is evident that, subject to this first condition, every different value of $\lambda$ will give a different value of $\Psi$, and that of these values of $\Psi$ that which gives the least value of P, and which corresponds to a *positive* value of $\lambda$ not greater than $\alpha$, will be the true angle of rupture, on the hypothesis of a mathematical adjustment of the surfaces of the voussoirs to one another. To determine this minimum value of P, in respect to the variation of $\Psi$ dependent on the variation of $\lambda$, or

of $p$, let it be observed that $\lambda$ does not enter into equation (8); let that equation, therefore, be differentiated in respect to P and $\Psi$, and let $\frac{d\,\mathrm{P}}{d\,\Psi}$ be assumed $= 0$, we shall thence obtain the equation

$$\sec.^2\Psi = \frac{3\,\alpha\,(\alpha+2)}{\frac{6\,\mathrm{P}}{r^2}+\alpha^2\,(2\,\alpha+3)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (28)$$

whence observing that $\sin.\,2\,\Psi = \frac{\tan.\,\Psi}{\sec.^2\Psi} = \left\{\frac{\frac{6\,\mathrm{P}}{r^2}+\alpha^2\,(2\,\alpha+3)}{3\,\alpha\,(\alpha+2)}\right\}\tan.\,\Psi$, we obtain by elimination in equation (8),

$$\sin.\,2\,\Psi - 2\,\Psi = \frac{4\,\mathrm{Y}}{\alpha\,(\alpha+2)\,r^2} - 2\,\Theta \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (29)$$

from which equation $\Psi$ may be determined. Also by equation (28)

$$\frac{\mathrm{P}}{r^2} = \tfrac{1}{6}\left\{3\,\alpha\,(\alpha+2)\cos.^2\Psi - \alpha^2\,(2\,\alpha+3)\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (30)$$

and by eliminating sec. $\Psi$ between equations (28) and (29) and reducing

$$\frac{p}{r} = (1+\lambda)\cos.\,\Theta = \frac{r^2}{\mathrm{P}}\left\{\sqrt{\alpha\,(\alpha+2)\left\{\frac{2\,\mathrm{P}}{r^2}+\alpha^2\,(\tfrac{2}{3}\alpha+1)\right\}} - \alpha\,(\tfrac{1}{3}\alpha^2+\alpha+1)\cos.\,\Theta - \frac{\mathrm{Y}\,x}{r^3}\right\} \quad \ldots\ldots \quad (31)$$

The value of $\lambda$ given by this equation determines the actual direction of the line of resistance through the key-stone, on the hypothesis made, only in the case in which it is a *positive* quantity and not greater than $\alpha$; if it be negative, the line of resistance passes through the bottom of the key-stone, or if it be greater than $\alpha$, it passes through the top.

It will be observed that the equation (11) determines the angle $\Psi$ of rupture in terms of the load Y, and the horizontal distance $x$ of its centre of gravity from the centre C of the arch, its radius $r$, and the depth $\alpha r$ of its voussoirs; moreover, that this determination is wholly independent of the angle of the arch, and is the same whether its arc be the half or the third of a circle; also, that if the angle of the semi-arch be less than that given by the above equation as the value of $\Psi$, there are no points of rupture, such as they have been defined, the line of resistance passing through the springing of the arch and *cutting* the intrados there.

The value of $\Psi$ being known from this equation, P is determined from equation (8), and this value of P being substituted in equation (6), the line of resistance is completely determined; and assigning to $\theta$ the value ACB (fig. 25), the corresponding value of $\rho$ gives us the position of the point Q, where the line of resistance intersects the lowest voussoir of the arch or the summit of the pier; moreover, P is evidently equal to the horizontal thrust on the top of the pier, and the vertical pressure upon it is the weight of the arch and load: thus all the elements which determine the equilibrium of the pier (see page 41) are known, and by substituting these in equation (4), page 46, we may determine the greatest height of a pier of given dimensions, or the least dimensions of a pier of a given height, which shall just sustain the proposed arch and its loading.

Thus every element of the theory of the arch and its abutments is involved in the solution in respect to $\Psi$ of

equation (11). Unfortunately this solution presents great analytical difficulties. It may be effected under the form of a series ascending by powers of $\alpha$, and the two first terms of that series have been given in the paper on the arch, already referred to, in the fifth volume of the Cambridge Philosophical Transactions.

The third and fourth terms of this series assume exceedingly complicated analytical forms. Happily, however, in the failure of any direct means of solution, there are various methods by which the numerical relation of $\Psi$ and Y in equation (11) may be arrived at indirectly. Among them one of the simplest is this :—

Let it be observed that that equation is readily soluble in respect to Y; instead then of determining the value of $\Psi$ for an assumed value of Y, determine conversely the value of Y for a series of assumed values of $\Psi$. Knowing the distribution of the load Y, the values of $x$ will be known in respect to these values of $\Psi$, and thus the values of Y may be numerically determined and may be tabulated; from such tables may be found by inspection values of $\Psi$ corresponding to given values of Y.

The arch and its loading have here been supposed to be given, and from these it has been sought to determine the conditions of its stability. In the majority of cases, however, the loading, placed within the power of the engineer, has to be *determined;* the problem being so to load the arch as to be *assured* of its stability. This is a problem of comparatively little difficulty; we have only to assume a value of $\Psi$, and from equation (11) to determine the corresponding load. All the circumstances

of the equilibrium of the arch thus loaded will then be known, and we may so determine the dimensions of its piers as to give it any required stability.

The values of $\Psi$, P, and $r$, are completely determined by equations (29, 30, 31), and all the circumstances of the equilibrium of the circular arch are thence known, on the hypothesis, there made, of a true mathematical adjustment of the surfaces of the voussoirs to one another; and although this adjustment can (see page 14) have no existence in practice where the voussoirs are put together without cement, yet may it obtain in the cemented arch. The cement, by reason of its yielding qualities when fresh, is made to enter into so intimate a contact with the surfaces of the stones between which it is interposed, that it takes, when dry, in respect to each joint, (abstraction being made of its *adhesive* properties,) the character of an exceedingly thin voussoir, having its surfaces mathematically adjusted to those of the adjacent voussoirs; so that if we imagine, not the adhesive properties of the cement of an arch, but only those which tend to the more uniform diffusion of the pressures through its mass, to enter into the conditions of its equilibrium, these equations embrace the entire theory of the cemented arch. The hypothesis here made probably includes all that can be relied upon in the properties of cement as applied to large structures.

An arch may FALL either by the sinking or the rising of its crown. In the former case, the line of resistance passing through the top of the key-stone is made to cut the extrados beneath the points of rupture; in the latter,

passing through the bottom of the key-stone, it is made to cut the extrados between the points of rupture and the crown.

In the first case, the values of X, Y, and P, being determined as before, and substituted in equation (6), and $p$ being assumed$=(1+\alpha)\ r$, the value of $\theta$, which corresponds to $\rho=(1+\alpha)\ r$, will indicate the point at which the line of resistance cuts the intrados. If this value of $\theta$ be less than the angle of the semi-arch, the intersection of the line of resistance with the extrados will take place above the springing, and the arch will fall.

In the second case, in which the crown ascends, let the *maximum* value of $\rho$ be determined from equation (6), $p$ being assumed$=r$; if this value of $\rho$ be greater than R, and the corresponding value of $\theta$ less than the angle of rupture, the line of resistance will cut the extrados at the point indicated by this value of $\theta$; the arch will there open at the intrados, and it will fall by the descent of the crown. If the load be collected over a *single* point of the arch, the intersection of the line of resistance with the extrados will take place between this point and the crown; it is that portion only of the line of resistance which lies *between* these points which enters therefore into the discussion. Now if we refer to page 50, it will be apparent that in respect to this portion of the line, the values of X and Y in equations (5) and (6) are to be neglected; the only influence of these quantities being found in the value of P.

*Example.* — Let a circular arch (fig. 26) of equal

voussoirs have the depth of each voussoir equal to $\frac{1}{10}$th the diameter of its intrados, so that $\alpha = \cdot 2$, and let the load rest upon it by three points A, B, D of its extrados, of which A is at the crown and B, D are each distant from it 45°, and let it be so distributed that $\frac{3}{8}$ths of of it may rest upon each of the points B and D, and the remaining $\frac{1}{4}$ upon A; or let it be so distributed within 60° on either side of the crown as to produce the same effect as though it rested upon these points.

Then assigning one half of the load upon the crown to each semi-arch, and calling $x$ the horizontal distance of the centre of gravity of the load upon either semi-arch from C, it may easily be calculated that $\frac{x}{r} = \frac{3}{4}$ sin. 45 = ·5303301. Hence it appears from equation (15) that no loading can cause the angle of rupture to exceed 65°. Assume it to equal 60°; the amount of the load necessary to produce this angle of rupture, when distributed as above, will then be determined by assuming in equation (11), $\Psi = 60°$, and substituting for $\alpha = \cdot 2$ and for $\frac{x}{r} =$ ·5303301. We thus obtain $\frac{Y}{r^2} = \cdot 0138$. Substituting this value of $\frac{Y}{r^2}$, and also the given values of $\alpha$ and $\Psi$ in equation (9), and observing that in this equation $\frac{p}{r}$ is to be taken $= 1 + \alpha$ and $\Theta = 0$, we find $\frac{P}{r^2} = \cdot 11832$. Substituting this value of $\frac{P}{r^2}$ in the equation (6), we have for the final equation to the line of resistance beneath the points B and D

$$\rho = r \frac{\cdot 2426 \text{ vers. } \theta + \cdot 1493}{\cdot 0138 \text{ sin. } \theta + \cdot 1183 \text{ cos. } \theta + \cdot 22\,\theta \text{ sin. } \theta}.$$

If the arc of the arch be a complete semicircle (fig. 26), the value of $\rho$ in this equation corresponding to $\theta = \frac{\pi}{2}$ will determine the point Q, where the line of resistance intersects the abutment; this value is $\rho = 1 \cdot 09\ r$.

If the arc of the arch be the third of a circle (fig. 27), the value of $\rho$ at the abutment is that corresponding to $\theta = \frac{\pi}{3}$; this will be found to be $r$, as it manifestly ought to be, since the points of rupture are in this case at the springing.

In the first case the volume of the semi-arch and load is represented by the formula

$$r^2 \left\{ (\tfrac{1}{2}\alpha^2 + \alpha)\tfrac{\pi}{2} + \frac{Y}{r^2} \right\} = \cdot 3594\ r^2,$$

and in the second case by

$$r^2 \left\{ (\tfrac{1}{2}\alpha^2 + \alpha)\tfrac{\pi}{3} + \frac{Y}{r^2} \right\} = \cdot 2442\ r^2.$$

Thus, supposing the pier to be of the same material as the arch, the volume of its material, which would have a weight equal to the *vertical* pressure upon its summit, would in the first case be $\cdot 3594\, r^2$, and in the second case $\cdot 2442\, r^2$, whilst the *horizontal* pressures P, would in both cases be the same, viz. $\cdot 11832\, r^2$; substituting these values of the vertical and horizontal pressures on the summit of the pier, in equation (4), page 46, and for $k$ writing $K - (\rho - r)$ we have in the first case

$$H = \frac{\cdot 3594\ (2\ K - \cdot 09\ r)\ r^2}{\cdot 11832\ r^2 - 2\ K^2}\ ; \quad \ldots\ldots \quad (32)$$

and in the second case

$$H = \frac{\cdot 4884\ K\ r^2}{\cdot 11832\ r^2 - 2\ K^2}\ ; \quad \ldots\ldots\ldots \quad (33)$$

where H is the greatest height to which a pier, whose width is 2 K, can be built so as to support the arch.

If $2\ K^2 - \cdot 11832\ r^2 = 0$, or $K = \cdot 243\ r$, then in either case the pier may be built to any height whatever, without being overthrown. In this case the breadth of the pier will be nearly equal to $\frac{1}{4}$th of the span. The dotted lines in the figure show the dimensions of such a pier.

The height of the pier being *given*, (as is commonly the case,) its breadth, so that the arch may just stand firmly upon it, may readily be determined. As an example, let us suppose the height of the pier to equal the radius of the arch. Solving equations (32) and (33) in respect to K, we shall then obtain in the first case $K = \cdot 1489\ r$, and in the second $K = \cdot 15\ r$.

If the span of each arch be the same, and $r_1$ and $r_2$ represent their radii respectively, then $r_1 = r_2$ sin. 60°; supposing then the height of the pier in the second arch to be the same as that in the first, viz. $r_1$, then in the second equation we must write for H, $r_2$ sin. 60°. We shall thus obtain for K the value $\cdot 14\ r_2$.

The piers shown by the dark lines in the figures 26 and 27 are drawn to a scale, and are of such dimensions as just to be sufficient to sustain the arches which rest

upon them, and their loads, both being of a height equal to the radius of the semicircular arch. It will be observed, that in both cases the load $Y = \cdot 0138\, r^2$, being that which corresponds to the supposed angle of rupture 60°, is exceedingly small.

Let us next take the example of a Gothic arch, and let us suppose, as in the last examples, that the angle of rupture is 60°, and that $\alpha = \cdot 2$; but let the load in this case be imagined to be collected wholly over the crown of the arch, so that $\frac{x}{r} =$ sin. 30°. Substituting in equation (11), 30° for $\Theta$, and 60° for $\Psi$, and ·2 for $\alpha$, and sin. 30° for $\frac{x}{r}$, we shall obtain the value 1·3101 for $\frac{Y}{r^2}$; whence by equation (9) $\frac{P}{r^2} = \cdot 67008$, and this value being substituted, equation (6) gives 1·048 for the value of $\rho$ when $\theta = \frac{\pi}{2}$. We have thus all the data for determining the width of a pier of given height, which will just support the arch. Let the height of the pier be supposed, as before, to equal the radius of the intrados; then since the weight of the semi-arch and its load is $1 \cdot 54\, r^2$, and the horizontal thrust $\cdot 67008\, r^2$, the width 2 K of the pier is found by equation (4) to be $\cdot 422\, r$.

Fig. 28 represents this arch; the square, formed by dotted lines over the crown, shows the dimensions of the load of the same materials as the arch, which will cause the angle of the rupture to become 60°; the piers are of the required width $\cdot 422\, r$, such that when their height is equal to AB, as shown in the figure, and the arch bears

this insistent pressure, they may be on the point of overturning. The great amount of the load Y, which in this arch corresponds to an angle of rupture of 60°, shows that angle to be much less than 60° in the great majority of the cases which are offered in the practice of the Gothic arch.

It is not possible, within the limits necessarily assigned to theoretical investigations in a work like this, to enter further upon the discussion of those questions whose solution is involved in the equations which have been given; these can, after all, become accessible to the *general* reader only when tables shall be formed from them.

The equilibrium of the arch is a particular case of the equilibrium of a system of bodies in contact. The conditions of the equilibrium of such a system will be found discussed under a general form in the APPENDIX.

H. MOSELEY.

King's College,
November 5th, 1839.

# APPENDIX

TO

PROFESSOR MOSELEY'S THEORY OF THE ARCH.

## GENERAL CONDITIONS OF THE EQUILIBRIUM OF A SYSTEM OF BODIES IN CONTACT.

1. LET a continuous mass, to which are applied certain forces of pressure, be supposed to be intersected by a *plane* whose equation is

$$z = Ax + By + C \quad . \quad . \quad .

resultant force with its corresponding intersecting plane in every position, which, according to the assumed conditions, this last may be made to take up.

This surface is the SURFACE OF RESISTANCE.

If to the preceding conditions there be added this, that in each two consecutive positions of the intersecting plane the corresponding resultants shall intersect, the surface of resistance will resolve itself into a line, which is the LINE OF RESISTANCE.

Differentiating on this hypothesis the equation (III,) in respect to A, B, C, we have

$$\left.\begin{aligned}
&\left\{\frac{d\left(\frac{M_1}{M_3}\right)}{dA}dA+\frac{d\left(\frac{M_1}{M_3}\right)}{dB}dB+\frac{d\left(\frac{M_1}{M_3}\right)}{dC}dC\right\}z+\left\{\frac{d\left(\frac{N_3}{M_3}\right)}{dA}dA+\frac{d\left(\frac{N_2}{M_3}\right)}{dB}dB+\frac{d\left(\frac{N_2}{M_3}\right)}{dC}dC\right\}=0\\
&\left\{\frac{d\left(\frac{M_2}{M_3}\right)}{dA}dA+\frac{d\left(\frac{M_2}{M_3}\right)}{dB}dB+\frac{d\left(\frac{M_2}{M_3}\right)}{dC}dC\right\}z+\left\{\frac{d\left(\frac{N_1}{M_3}\right)}{dA}dA+\frac{d\left(\frac{N_1}{M_3}\right)}{dB}dB+\frac{d\left(\frac{N_1}{M_3}\right)}{dC}dC\right\}=0
\end{aligned}\right\}\quad\text{(IV.)}$$

Eliminating $z$

$$\left\{\frac{d\left(\frac{M_2}{M_3}\right)}{dA}dA+\frac{d\left(\frac{M_2}{M_3}\right)}{dB}dB+\frac{d\left(\frac{M_2}{M_3}\right)}{dC}dC\right\}\left\{\frac{d\left(\frac{N_2}{M_3}\right)}{dA}dA+\frac{d\left(\frac{N_2}{M_3}\right)}{dB}dB+\frac{d\left(\frac{N_2}{M_3}\right)}{dC}dC\right\}$$

$$-\left\{\frac{d\left(\frac{M_1}{M_3}\right)}{dA}dA+\frac{d\left(\frac{M_1}{M_3}\right)}{dB}dB+\frac{d\left(\frac{M_1}{M_3}\right)}{dC}dC\right\}\left\{\frac{d\left(\frac{N_1}{M_3}\right)}{dA}dA+\frac{d\left(\frac{N_1}{M_3}\right)}{dB}dB+\frac{d\left(\frac{N_1}{M_3}\right)}{dC}dC\right\}=0\ldots\text{(V.)}$$

From the elimination of A, B, and C, between the five equations (I, II, III, V), will result the two equations to the LINE OF RESISTANCE; and from the elimination of the same three quantities between the five equations (II, III, IV),* the two equations to the LINE OF PRESSURE.

The inclination $\iota$ of the resultant pressure to a perpendicular to the intersecting plane, in any of its positions, may be determined, independently of the line of pressure, from the equation

$$\cos \iota = - \frac{AM_1 + BM_2 + M_3}{\{(A^2 + B^2 + 1)\,(M_1^2 + M_2^2 + M_3^2)\}^{\frac{1}{2}}}.$$

2. Let the mass be a PRISM whose axis is horizontal (fig. 3), and the forces applied to which are, its weight and certain pressures, P.

The relation of the forces which compose the equilibrium of the whole Prism, will then be the same with that of the forces impressed on any one of its sections perpendicular to the axis.

Let C B D, (fig. 3), represent any one of these sections. Suppose the mass to be intersected in any direction parallel to its axis by a plane, and let $N_1$ $N_2$ be the intersection of this plane with the section C B of the mass.

And let this intersecting plane in altering its position be supposed to remain always parallel to itself.

Take A $z$, the axis of $z$, perpendicular to $N_1$ $N_2$, and let it make an angle $\Theta$ with the vertical.

Let $MN_1 = y_1$, $MN_2 = y_2$, $AM = c$, $AK = k$.

* It will be observed that the condition (V) is included in these.

$\therefore M_1 = 0,\ M_2 = \Sigma P \sin.\Phi - \sin.\theta \int (y_1 - y_2)\, dC,\ M_3 = \Sigma P \cos.\Phi + \cos.\theta \int (y_1 - y_2)\, dC,$

$N_1 = \frac{1}{2}\cos.\theta \int (y_1^2 - y_2^2)\, dC + \sin.\theta \int C (y_1 - y_2)\, dC + \Sigma \pm P k \cos.\Phi,\quad N_2 = 0,\quad N_3 = 0.$

This hypothesis with regard to the position of the axis of $z$, and these substitutions being made, all the equations of condition vanish except equation I, the second of equations III, and the second of equations IV. These resolve themselves into the following:—

$$z = C \quad \ldots\ldots\ldots\

The second elimination is greatly simplified in the case in which P, $\Phi$, $k$ are independent of C. Since in this case, equation (3) gives

$$y-\tfrac{1}{2}(y_1+y_2)+(z-\mathrm{C})\tan.\Theta=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

If the intersections be supposed to be made horizontally, (fig. 4,) $\Theta$ must be assumed $=0$. If they be made vertically, (fig. 5,) $\Theta=\frac{\pi}{2}$. In the latter case, equation (5) gives $\mathrm{C}=z$.

The elimination of C between (3) and (2) is therefore the same as that between (1) and (2), and the line of pressure in this case coincides with the line of resistance.

3. Let the mass be of a trapezoidal form (fig. 6).

Let A B and C D be inclined to the axis of $z$ at angles $\alpha_1$, $\alpha_2$, and assume $\mathrm{C\,A}=a$; $\therefore\ y_1=a+z\ \tan.\ \alpha_1$, $y_2=z\ \tan.\ \alpha_2$.

$$\left.\begin{aligned} \text{Hence } \int(y_1^2-y_2^2)\,dz &= az\,(a+z\ \tan.\ \alpha_1)+\tfrac{1}{3}z^3(\tan.^2\alpha_1-\tan.^2\alpha_2)\\ \int(y_1-y_2)\,dz &= az+\tfrac{1}{2}z^2(\tan.\ \alpha_1-\tan.\ \alpha_2)\\ \iint(y_1-y_2)\,dz^2 &= \tfrac{1}{2}az^2+\tfrac{1}{6}z^3(\tan.\ \alpha_1-\tan.\ \alpha_2)\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (6)$$

Therefore by substitution in equation (4) we have for the equation to the line of resistance

$$y=\frac{\frac{1}{6}z^3\{\tan.\alpha_1-\tan.\alpha_2\}\{\tan.\alpha_1+\tan.\alpha_2-\tan.\Theta\}+\frac{1}{2}az^2\{\tan.\alpha_1-\tan.\Theta\}+z\{\sec.\Theta\Sigma\mathrm{P}\sin.\Phi+\frac{1}{2}a^2\}+\sec.\Theta\Sigma\pm\mathrm{P}k\cos.\Phi}{\frac{1}{2}z^2\{\tan.\alpha_1-\tan.\alpha_2\}+az+\sec.\Theta\,\Sigma\,\mathrm{P}\cos.\Phi} \quad (7)$$

This equation being of three dimensions in $z$, it follows that for *certain values* of $y$ there are three possible values of $z$. The curve has therefore a point of contrary flexure, and is somewhat of the form shown in the figure.

A SERIES OF PAPERS

ON THE

# FOUNDATIONS OF BRIDGES.

BY T. HUGHES, CIVIL ENGINEER.

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## No. 1.

THE foundations of a bridge, consisting properly of the underground work of the piers and abutments, must naturally claim, in a very eminent degree, the attention of the engineer or bridge architect.

The most refined elegance of taste, as applied in the architectural embellishment of the structure—the most scientific arrangement of the arches, and disposition generally of the superior parts of the work—and the most judicious and workmanlike selection, and subsequent combination of the whole materials composing the edifice, are evidently secondary to the grand object of producing certain firm and solid bases, whereon to carry up to any required height the various pedestals of support for the arches of the bridge.

Indeed, the necessity of firmness and solidity in the foundations will be deemed of importance just in pro-

portion to the intended extent and magnificence of the structure they are designed to support.

The distinction made by the celebrated and learned Leon Battista Alberti between the structure above ground and the foundations of any building is remarkably applicable to the case of bridges. He considers the foundation, not as a part of the structure itself, but as an artificial support on which the latter is to be placed; and justly observes, that if the natural site of a building consisted of rock, or other stratum equally hard with the material of which foundations are constructed, these would be unnecessary, and the building might be commenced and carried up without previous preparation of the bottom.

In founding bridges to be built over roads and railways, it seldom happens that much water is met with, and the simple expedient of piling, or using concrete, where the bottom is unsound, may readily be effected, and will usually ensure the stability of the superior work. The difficulty which sometimes will occur in dry and easily accessible cases of foundations will hereafter be discussed; the immediate object to which this paper will be principally confined being that of elucidating from practical example the most approved methods of founding the more difficult and important kinds of bridges, intended to carry roads of communication across canals and rivers. Although in such bridges the abutments may occasionally be founded on the dry land, and in some situations where the course of the river can be turned, a similar facility for founding the piers may be

obtained, yet many instances have occurred, and many will, no doubt, occur in future, where bridges must be built over deep and rapid rivers, the course of which can on no account be diverted, and where, consequently, all the difficulties of operating in the midst of water must be contended with and overcome.

Before attempting to classify and describe in detail the various methods at present in use for constructing the foundations of bridges under water, it may be interesting briefly to review some of the older and ruder contrivances of our ancestors. Treating on this subject, an article of great research and ability is given in the Encyclopædia Metropolitana, which describes four methods of laying foundations, by means of caissons, the oldest form of which is no more than a basket of strong construction, such as are made of the pliable boughs or branches of trees, which, being weighted with stones, are lowered to the bottom, and then filled with the same description of material, until raised to within a foot or sixteen inches of the lowest water surface. The application of ingenuity and science to works of this kind naturally led to the substitution of wooden chests, strongly hooped with iron, instead of the original baskets; and these wooden chests, being filled with a sufficient weight of masonry, were sunk to the bed of the river, as practised by Labelye and Mr. Mylne in the construction of Westminster and Blackfriars Bridges. It must not be supposed, however, that the idea of using caissons of a sufficient size to

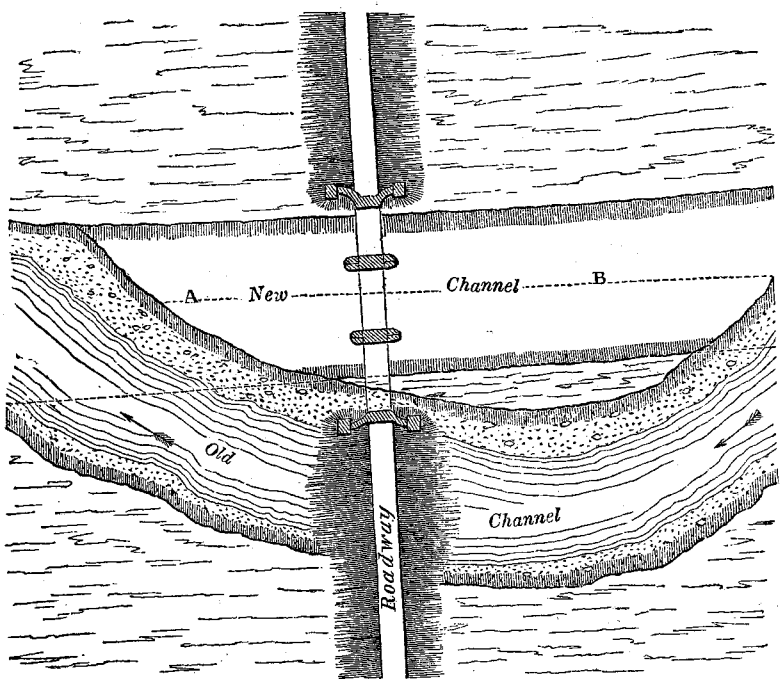
contain a whole pier was put in practice in the early days of bridge building. The caissons of basket-work here mentioned were of small dimensions, and several of them, according to their size, were required to compose the foundation of a pier or abutment. The first use also of wooden caissons was limited to those of five or six feet square, and it may therefore be ranked as a bold and most important advance of modern science to practise the expedient of constructing in a caisson above ground immense masses of masonry, weighing many hundred tons, and sinking them, still solid and united, down to the ground, through a considerable depth of water.

Considering only shallow rivers, it would seem that the rude caissons used by our ancestors effected tolerably well the purpose for which they were designed, but in rivers of magnitude, and particularly those where strong currents existed, it was found necessary to construct the piers on dry ground. The most obvious expedient for attaining this end is, in the first instance, to preserve a good and direct approach, leading to and over the bridge, and then to place the site, not over the river, but at some convenient distance from it, forming the piers in a range with the general direction of the stream, and at right angles with the approach alluded to, then turning the course of the river by a new channel through the water way of the bridge.

This will at once be understood by referring to the accompanying figure 1, which represents the plan of a

bridge to be built in a neck of land, formed by a bend of the river, which must afterwards be diverted from its

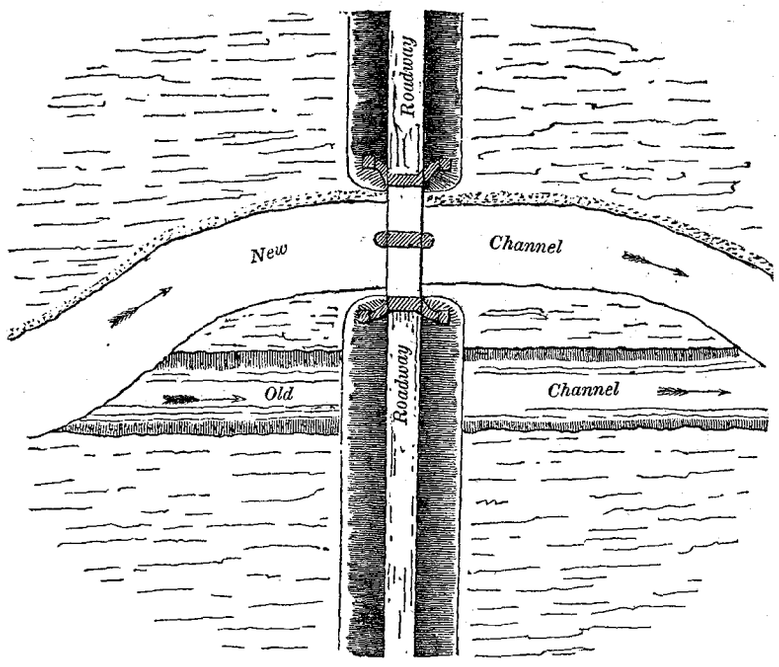
FIG. 1.




course in the direction of the dotted line AB, and the roadway can be readily embanked across the old channel, which will be laid dry. Even when the original course of the river is nearly straight, the expense of fixing the bridge in that situation might be attended with such serious difficulties, as to warrant the engineer in slightly turning the river, and thus have the advantage of building on a firm bottom at a very moderate cost, as compared with the other situation, and so avoid the danger and difficulty to which I allude; the accompanying sketch, fig. 2, shows the appearance of the diversion; the bridge to be built on dry land on either side of the old channel.

Such are the circumstances of position in which many

FIG. 2.




important bridges have been constructed, and as the execution of works, under similar circumstances, will in all probability prove of frequent occurrence to modern practitioners, it may be necessary, in a future page, to make some remark on the most practicable method of building the bridge, and effecting the diversion of the river in such situations.

This, it will obviously be seen, involves the necessity, during the period of getting in the foundations, of keeping them clear of water, which might drain into them from the adjacent river; and again, some skill and experience will naturally be required in the management of the river at the time when the new channel is prepared to receive it, otherwise dangerous consequences might arise from the overflow, and breaking down of the banks, or temporary dams or stanks.

All this will be treated of in future, whilst at present we confine ourselves to the kind of foundation adopted by the ancients for large bridges, built in the situation we have been describing.

This brings us to the subject of piles, which, when the bottom was very unsound, were driven down entirely over the site of the proposed piers. The most remarkable difference between the practice of the ancients and the moderns in founding on piles as a foundation, is that the former, instead of evenly cutting off the pile heads, and planking over them with a platform of timber, were accustomed to fill in between them, with a species of coarse concrete, called by the French *béton*, and which, being brought to a level surface, formed the bed on which the first course of stone was placed.

We believe that since some recent publications on the subject of limes and concrete, particularly that very excellent one by Colonel Pasley on calcareous mortars and cements, and another by Mr. Godwin on concrete, no apology is necessary for what some years since would have been pronounced the anachronism of ascribing the use of concrete to the ancients, and especially to the Romans. Both the authors above mentioned concur in admitting that the use of a concrete mixture, composed of lime and coarse gravel, is by no means an exclusively modern practice.

The piles for foundations of old bridges were distributed much in the same way as for modern structures, namely, three or four feet apart, and the extent to which

they were driven was regulated, as of course it ought to have been, by the nature of the ground bottom.

Another method of using piles, besides that of establishing a foundation for the masonry, was much practised by the ancients for building in water. This method is called by the French *encaissement*, and is still practised along the shores of the Mediterranean. The following is the system pursued. Main piles are driven in, with sheet-piling between them, secured and bound round with waling, as in the modern coffer-dam, but it must be understood that for the purpose of encaissement, only one row of piles is to be driven all round the space of the pier or abutment to be founded, and not a double row of piles to be filled with clay as in a coffer-dam. When the space to be occupied by the foundation has been entirely enclosed by this wooden encaissement, and the compressible material within the enclosure has been taken out, a mass of concrete or dry rubble stones is thrown into it, and the foundation is formed in this manner until the work reaches the level of the water. Foundations within encaissements should be allowed some time to settle, and become solid before the dressed masonry is laid in courses upon them.

In this manner many celebrated works were founded on the Continent, and the system was particularly approved of by Belidor, who much preferred it to that of constructing foundations entirely of dressed stones. Another great argument in favour of this system arises from the saving of coffer-dams, which are always found to be very tedious and expensive works.

As connected with the use of concrete during late years in the foundations of bridges, Mr. Peter Semple, architect of a bridge over the Liffey at Dublin, seems entitled to considerable credit, if not for the re-invention of this substance, at least for setting the example of using it on an extensive scale.

This author gives a design, which bears a precise resemblance to the form of encaissement above described. In a river six feet deep, for example, he proposes to drive sheeting piles, about ten feet in length, to a depth of four feet into the ground all round the site of each pier, and to fill the space or coffer thus formed with a bed of concrete, six feet in depth.

The top of the concrete being level with the surface of summer low water mark, he proposes thereupon to commence the masonry, and carry it up to springing height clear of the water.

Before concluding this brief glance at the practice of the ancients, it may be necessary to notice, that on examining the foundations of old bridges, particularly in this country, they are all found to be extremely massive, and the piers were even carried up above water, of a thickness quite incommensurate with the necessity of the case. On inspecting the masonry of their foundations, however, it is found that no very great attention has been paid to the regularity of courses, or to the perfection of beds and joints. Some of the strongest specimens of ancient masonry existing in this country consist of a kind of building little superior to rubble walling, with this most important qualification, that the

mortar is always of an excellent description, and in most cases, by no means inferior in hardness and cohesiveness to the stone itself.

It is no uncommon thing to witness, for instance, the face of old buildings, where the masonry has been indented and sensibly decomposed by the varied action of air, water, and frost, while the horizontal bands or joints of mortar project sensibly beyond the stone-work, clearly indicating that a greater resistance has here been offered than by the stone itself. The mortar of the ancients usually contained a great number of small stones or pebbles, some of them equal in size to a pigeon's egg, and was altogether of a much coarser description than that used in the present day. Another point of importance in the bridge architecture of the ancients is deserving of attention. For the most part, the bold designs of modern engineers, who devote their talents to the object of spanning the most majestic rivers with one vast arch, were preceded by structures, consisting of a long low series of culverts, hardly deserving the name of arches, with intervening piers, often of greater thickness than the span of the arches they were built to support. Bridges of this kind are worthy of no higher appellation than very thick walls, or embankments, perforated by a multitude of small openings, and, as compared with the structures of modern days, they are utterly contemptible and insignificant. But in founding the piers of such bridges a great deal of the difficulty attendant on the modern works was entirely avoided. We know that if a given weight be distributed among a great many points

of support, the pressure upon each point is proportionably diminished; and hence the twenty or thirty piers, which were sometimes built in the old bridges, had each to support very little more than its own weight.

It is easy from this consideration to discover one reason why the precaution of piling was so commonly dispensed with, although it would probably be erroneous to attribute it entirely to this circumstance. The ancients, in their bridges, throughout the whole structure, substituted quantity for quality, that is to say, huge masses of rough undressed masonry, or rubble, instead of the firm, compact, and elegant piers of modern bridges, which, above the bed of the river at all events, are invariably built with the most durable stone, of well-squared dressed ashlar fronts and suitable filling within.

The charge of clumsiness, however, conveys no reproach upon the judgment of an engineer, if it be confined to the work sunk into the ground under water, particularly as in such a situation it presents no obstruction to the free course of the river.

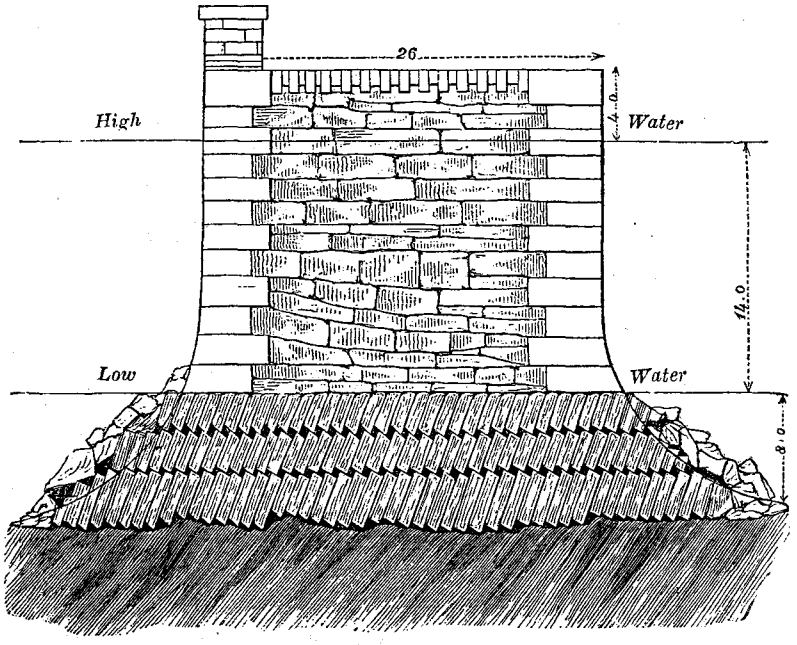
Indeed, it may often be most judicious, because most economical, to construct large and massive bases for the piers of a bridge to be raised upon, instead of building up regularly dressed courses of masonry from the very bottom, within a coffer-dam.

As this practice has been very successfully adopted in some projecting sea piers of modern construction, as well as of great magnitude, it may not be uninteresting

to describe them as practical examples, which may be safely followed under corresponding circumstances. The first work of the kind I shall describe was projected by Mr. Telford, and executed under the superintendence of Mr. David Henry, at Ardrossan Harbour, in Ayrshire, N. B.; and as the mass of stones used in the foundation was there set in tolerably regular order under water, without the aid of coffer-dam, or caisson of any kind, there can be no doubt of the same system being equally practicable in many cases of bridge foundations.

The stones at Ardrossan were of very large superficial dimensions, varying from six to ten feet long, and three to five feet wide; they were first held fast by an implement, technically called nippers or devil's claws, and were then lowered by a crane through a depth of six or eight feet of water on to a hard and solid foundation. The blocks were placed end to end, the position of the last stone lowered being found by probing with a slight iron rod; and as soon as each stone was in its place longitudinally, the claws were disengaged, and the stone allowed to rest upon the course below, as seen in fig. 3. The courses were continued entirely through the whole thickness of the pier; and when a sufficient number had been laid to bring the work up to the height of low water spring tides, the whole breadth was levelled, and all the unequal projections chipped off, in order to prepare a bed for the first course of dressed masonry. The work then proceeded in the regular manner, consisting of alternate headers and stretchers of properly squared

FIG. 3.



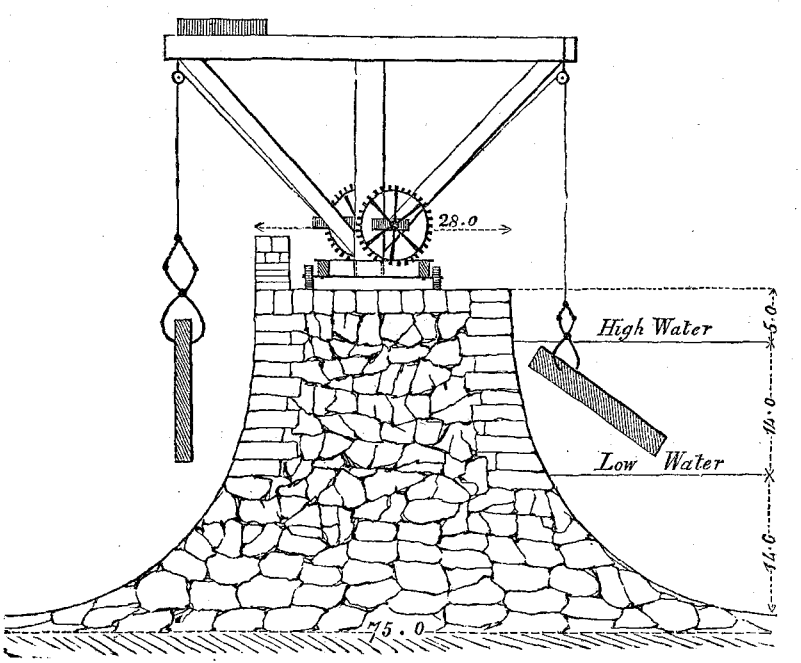

ashlar in front, with dry stone hearting of squared scapple dressed rubble inside, and in this way was carried up to the full height required.

When the writer visited this work, in the year 1818, it had been advanced a considerable distance into the sea; and although parts of it had been exposed to some very heavy storms, neither flaw nor settlement could be discovered in any part of this excellent piece of dry-built masonry.

From an account of some foundations similar to that described above in the recently published life of Mr. Telford, it may be seen that the practice has been much more extensively adopted, and a far bolder attempt carried out by Mr. Gibb, of Aberdeen, than the one acted upon in the other work at Ardrossan. The pier at Aberdeen is extended into the sea, with a breadth at

the base of seventy-five feet, the bottom consisting entirely of irregularly shaped masses of stone, which having been conveyed to the spot in boats, were tumbled in by chance to the depth of ten or twelve feet. In the drawings composing the Atlas, which accompanies the life of Mr. Telford, the low water mark is shown about fourteen feet above the bottom, and in the narrative of this work by Mr. Gibb, he states, that the bottom under the foundation is nothing better than loose sand and gravel, and that the front ashlar commences at about one foot under low water mark, and is carried up to the top of the pier, which the drawing shows to be about thirty-three feet in height from the bottom to the top. The rise of the tide is shown to be fourteen feet, the breadth of the pier twenty-eight feet, the sides carried up with a slope inwards. Fig. 4. describes the method adopted by Mr. Gibb.

PIER AT ABERDEEN. FIG. 4.



The last description of foundation which we shall describe as practicable in many situations, where the necessary funds cannot be obtained for raising the piers of a bridge within a coffer-dam, or on a caisson bottom, was practised at Inverness, under the superintendence of the writer, agreeably to instructions given by Mr. Telford. In order that the nature of this work may be understood, it must be explained that the old harbour of Inverness is situate on a part of the river Ness, about a quarter of a mile from its entrance into the firth or arm of the sea, with which it communicates. The river has a considerable fall from the old harbour towards the sea, and the tide does not rise within several feet of the same height at the former place, as it does lower down. The consequence was, that the larger description of vessels which frequented the port could navigate the river only at high water of spring tides; and as the trade increased in common with that of many other ports, during the late war, this inconvenience was most sensibly felt.

The town funds were inadequate to defray the expense of deepening the river by bringing up the dead level from the sea, and afterwards building from the bottom a suitable breast work for unloading at the harbour. The expedient eventually decided upon was that of building a pier or landing wharf at some distance down the river, at a place where the water was already tolerably deep, and at which large vessels, passing up the river, might discharge a part of their cargo, and then, after being thus lightened, and rendered capable of floating in a less depth of water, pass up with the remainder of their

cargo into the more convenient accommodation afforded by the harbour.

The usual and modern practice of founding works of this kind is by coffer-dam, but the low state of the funds applicable to the work rendered it impossible to carry into effect so expensive an operation. In this extremity, Mr. Telford was consulted, and he, considering all the circumstances of the case, and particularly the great necessity for practising economy, gave instructions for a mode of putting in the foundations, which it may be useful to describe in detail. At the site fixed upon for the intended pier, the depth of water, at the lowest spring tides, was never less than four feet, and at ordinary low water five or six feet; the bottom a very hard gravel, united with clay. The whole length of the breast work was about one hundred and sixty feet, and throughout this distance the bottom was dredged out, to the width of eight feet, and depth of two feet, to receive the masonry.

A simple system of piling was however driven previous to founding the masonry. The piling consisted of two bearing piles, twelve feet long, and eight inches diameter, driven down at intervals of twenty feet; and across the heads of these piles, and level with low water mark, cross pieces of elm planking twelve feet long, three inches thick, and one foot wide, were fastened with trenails. On the top of these were laid longitudinal half timbers, one foot wide, and six inches deep, secured to the cross pieces and bearing piles by rag bolts, driven into each pile head.

The accompanying sketches, figs. 5 and 6, will amply illustrate the forms and disposition of the timber work in the foundation. In addition to the bearing piles,

FIG. 5.

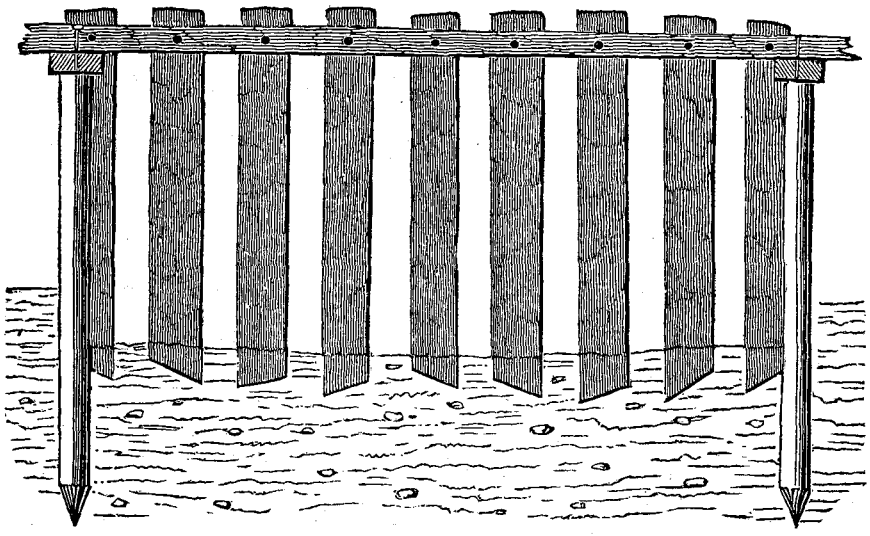
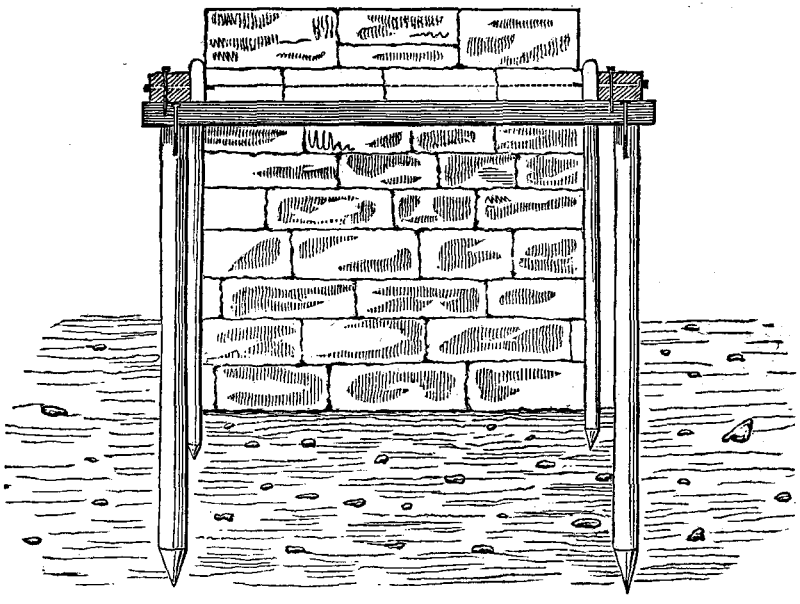


FIG. 6.



a row of timber slabs, of inferior quality, was also driven down a few inches into the bottom, at intervals of about ten or twelve inches; these had a spike driven through

them, near their heads, and into the longitudinal logs of half timbers; they were merely to answer the purpose of guide timbers, to set the stones by, and to determine the guage or breadth of the work, and were afterwards removed.

The bottom on which the pier was to be founded being now made as level as possible by means of dredging with the common bag and spoon apparatus, the stones were brought to the place in boats, and lowered by a crane, in such a way that as soon as each stone was placed in its proper position the *lewis* could be withdrawn without difficulty.

This will be understood on referring to fig. 7, which represents the lewis, fixed in a stone, ready prepared for being lowered through the water into the foundation. The lewis consisted of two pieces of iron B and D, and in order to use it a part of the stone must be cut out, sufficiently wide at top to receive the base of the part B, the base of the opening of the stone being equal to the united width of D and B; A is the chain suspended from the arm of the crane,[1] and E a small rope or string, of which the

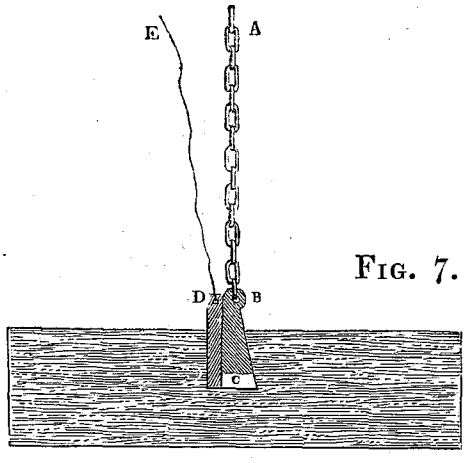



FIG. 7.

[1] It is quite evident that by any other mode of suspending the stones excepting that of the lewis, which could be disengaged under water, even an approximation to a close joint could never have been effected in the situation now described.

end is kept above water, to pull out the rectangular part D of the lewis.

It is easy to see the method of using this instrument: the piece B is first inserted, and D is then put in to secure it, when it is evident that the heavier the stone may be, provided it be strong enough, the more securely will it be held by the lewis when suspended from the crane. Conceive the stone now to have been lowered through the water, and carefully laid in its proper place in the foundation; the chain from the barrel of the crane is then loosened, and the part B of the lewis being slightly knocked with an iron rod from above, is easily made to drop down into the vacant space C. It is evident that the fastening piece D will then be loose, because between this and B there is a space left equal to the difference between the base of B, and the base of the opening in the stone. D may therefore be drawn out by means of the string E, and B will readily follow on pulling the chain A, and the lewis is again ready to be inserted in another stone.

All the front stones of the foundation were laid with a lewis of this kind, as well as the backing of squared stones, which were previously scapple-dressed at the quarry. The whole of the stones in any one course, for the length of the pier, were laid of equal thicknesses; they ranged from four to seven feet long, and from three to four feet wide. As soon as one course was complete another was laid, and the length of each stone being marked on the longitudinal beams above the piling, it was easy to set them so as to break bond, and the whole

process of thus building under water was effected with the utmost regularity, and with less difficulty than could have been anticipated by the most sanguine advocates of the plan.

When all the building was carried up as high as the surface of the lowest water mark of a spring tide, any irregularity on the top was taken off, and the whole surface carefully levelled, and on it the ashlar masonry was commenced and carried up with a vertical batter. This work consisted of stones with picked fronts and chisel-draughts round the edges, the ends, beds, and face, properly squared. The backing was of good common rubble, and the whole being raised to three feet above the highest spring tides, was finished off with a heavy coping, properly dowelled, cramped, and secured with lead.

This work, from its situation, is called the Thorn Bush Pier; the date of its construction was 1815, and up to the present time no appearance of failure or imperfection has been observed.

Another proof is thus afforded that foundations of magnitude and importance may be laid under water without resorting either to coffer-dams or caissons. It is important, however, that the engineer, before determining on adopting a plan of this bold and somewhat hazardous nature, should be well satisfied of the firmness and solidity of the bottom on which his work is to be founded. Very hard gravel, some of the clays, and rock, are excellent as foundations; but these, amongst other kinds, will be particularly discussed, when, in a future paper, we shall treat of the probable causes of failure

of settlement, and giving way of the piers and abutments of bridges.

## No. 2.

WHEN an engineer decides on founding the piers of a bridge within a coffer-dam, it has been a very general practice to leave the form and construction of the dam entirely to the contractor, on whom rests the responsibility of devising the proper means for keeping out the water while such a foundation is being constructed, as may have been described in the specification, or afterwards directed by the engineer, when he becomes fully acquainted with the nature of the ground to be built upon.

This practice of leaving undefined the mode of constructing the coffer-dam, and thus relying upon the contractor's judgment and experience in all the important details relating to the length and scantling of the piling, the extent of driving, and the thickness of the clay puddle between the rows, has been followed, we believe, almost universally by the railway engineers in their specifications for bridges, where it was likely that a coffer-dam would be required.

Hence, the young engineer must look to other sources than the ordinary modern practice of engineering for information on the subject of coffer-dams, and probably to no work in this country, or indeed in Europe, can he turn with more certainty of deriving the best practical knowledge than to those excellent dams constructed by Mr. Thomas Rhodes at the St. Katherine's Dock, under the direction of Mr. Telford; and those more recently

constructed under the superintendence of Mr. Walker, the one at Blackfriars Bridge, rendered necessary during the repair of the foundations, and the other at the site of the New Houses of Parliament at Westminster.

Mr. Walker, who, from his position as President of the Institution of Civil Engineers, and as the successor of Mr. Telford in that honourable station, may be considered the official head of his profession in this country, has not thought it beneath his dignity to furnish the contractor with drawings, and a detailed specification of the form and entire construction of the coffer-dam. On these, and those constructed by Mr. Rhodes, the young engineer may safely depend in all similar works with which he may be engaged. We shall, however, in our next, give instructions for constructing a dam to stand against forty feet of water.

The practice of employing caissons appears, during late years, gradually to have been given up, and few modern examples of their use can be found; Mr. Walker, we believe, used them very successfully at the Vauxhall Bridge. We are not aware that Mr. Telford, in all his practice, ever employed one; his preference, in the case of foundations in deep water which could not be diverted, being decidedly given to the system of enclosing the site within a coffer-dam. The best account of caissons, particularly those used by Mr. Mylne at Blackfriars Bridge, will probably be found in the article before referred to in the Encyclopædia Metropolitana.

In many tide rivers, where the depth at low water is not considerable, the foundations have been constructed

without coffer-dams, the men working only during certain intervals, immediately before and after flood or low water. This plan of proceeding cannot very successfully be adopted, however, unless the bed of the river be of rock, or other safe material for placing the foundations upon. The instance already given of an extensive harbour wall being founded under water, even where it was necessary to excavate the bed of the river under the site of the wall, deep enough for the foundation to rest upon, is an exception to the usual plan of coffer-damming, but might with great advantage be practised at any time in similar situations, where the depth of water does not exceed seven or eight feet. The method of encaissement filled up with concrete, as described in a former paper, affords great facilities for establishing a foundation in any depth of water not exceeding ten feet.

Probably the most difficult kind of foundation which the engineer can have to contend with is to be found in the flats of Norfolk, and in the extensive fens of Lincolnshire and Cambridgeshire. Throughout these districts the alluvial deposit is of great depth, and the flow of the rivers extremely sluggish in their progress towards the sea; circumstances which have induced the formation of strata which are remarkably treacherous and unsafe as foundations. The strata consist of soft and irregular deposits of fine sand, gravel, and mud, alternating with beds of peat and bog, and occasionally mixed with portions of clay. Sometimes these varieties of soil occur in patches; for example, seven or eight feet

square of gravel, surrounded perhaps by peat, or instead of the gravel fine sand is often found in a similar situation, while perhaps, close at hand, layers of peat have been formed in tolerable order; above them sand may occur, succeeded by gravel, and afterwards peat again. At other places, perhaps also within a short distance, the soil will consist of gravel, peat, and sand, all mixed up together, sometimes to the depth of thirty or forty feet before a foundation of clay or firm gravel can be met with. Under these circumstances, if the depth of water be considerable, and the structure heavy, the engineer's difficulty will be far from trifling, unless indeed he can command an unlimited purse, in which case it would be paying a poor compliment to engineering science to dwell upon the *difficulty* of executing any work. This word, in the vocabulary of an engineer, is only with propriety applied to the task of constructing his works within reasonable limits of expense, and he commits a libel on the judgment and talents of the profession who talks of the difficulty experienced by an engineer whose expenditure has been equally unlimited with his command of capital.

We shall now attempt an outline of the plan which an engineer should follow in preparing his foundations, where the ground is of the nature we have been describing. Suppose the bridge to be built over a river with five feet depth of water at the lowest summer floods, that the breadth of the waterway is great, and that no possibility exists of laying dry the bed of the river by turning its course: in such a situation it is

certain no method can be so good as that of driving a coffer-dam all round the space to be occupied by the piers, as well as the abutments. The depth of water being five feet, and of the unsound bottom say twenty-five feet, the piles can in no case be less than forty-five feet long, and driven into the solid ground as far as they will go, which may probably be from eight to ten feet. For such a depth of water, a double dam with three rows of piles will be necessary. The coffers between the rows of piles should be six or seven feet apart, and filled with a retentive clay puddle: the next operation is to get out all the water and soft material from within the coffer-dam. To effect this, various contrivances will present themselves; among which, in almost any case, a steam engine must be in requisition, for pumping out and clearing the space of water, and for keeping it dry during the progress of the building. The earth may be drawn from within the coffer-dam by means of windlasses, to be erected on a temporary stage or platform placed across the dam; and it will be found an excellent expedient to use buckets provided with bottoms which will open when they are drawn up by the windlasses, and thus discharge their contents into barges placed alongside the coffer-dam.

In this way the space to receive the piers and abutments may be expeditiously taken out, and as soon as the soft material has been removed, should the bottom be found to consist of hard gravel or clay, nothing more will be necessary than to commence the building after sinking into it about two feet. It is always good prac-

tice to lay broad and large bedded stones on the foundations, in order to give as much base, with as few joints as possible, for the superstructure to rest upon, whether it be intended to build the piers with brick or stone: it is very important, if practicable, to obtain stones of large superficies for the first and second courses of the building. The foundations both of piers and abutments, when of such a depth as we have been describing, should be carried up with offsets, so that each course for the first four feet in height from the bottom should project all round at least six inches beyond the course immediately above. The offsets may then be discontinued, and the building carried on according to the particular form of the design. If, after excavating within the coffer-dam to the depth above described, the bottom should still appear doubtful and unsound, a number of piles should be driven all over the space to be occupied by the building, and extending about a foot beyond it in every direction.

The piles may be eight or nine inches square, or round timber of this diameter, and should be driven in rows from two to three feet apart, as far as they can be forced into the solid ground, that is, until each pile will not sink more than a quarter of an inch with twenty blows from a ram of fifteen hundred pounds weight or thereabouts. Some practical men recommend that the piles should be driven first in the centre, and last at the ends and sides, alleging as a reason, that by adopting an opposite practice the ground becomes so solid as to prevent the piles from sinking in the centre;

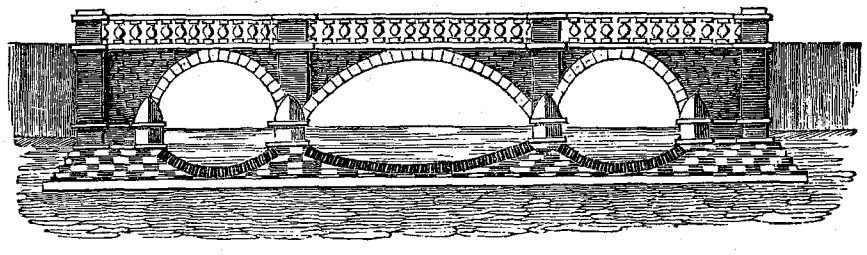
the writer however prefers the method of driving the outer piles first, and in this way consolidating the ground, with as little delay and cost as possible. It is true, by driving from the outside, and closing in towards the centre, that the last piles are prevented from being forced in so deep, but this only renders it necessary to use shorter piles, and the foundation thus becomes quite as solid as if the ground were forced outwards from the centre, and long piles driven in over the whole space. With respect to the description of timber which should be used for these piles, provided it be sound and fresh, the particular kind is of little consequence. Mr. Telford frequently specified Scotch fir; and the inferior kinds of timber, such as beech, elm, and birch, will answer very well, as they stand the driving better than fir, which, if not hooped, is apt to splinter, or the head become besomy and brushy, a circumstance which adds to the difficulty of driving.

As soon as the piles have all been driven, their heads should be cut off quite level, and about a foot in depth being excavated between them, the space up to the level of the pile heads should be filled with broken stones, grouted with good lime and sharp sand. At this stage, a platform of oak, beech, or elm plank, from four to six inches in thickness, should be laid across the pile heads, and secured to them either with spikes, bolts, or trenails of hard wood. This first covering of planking is usually succeeded by another of equal thickness laid across, the whole closely jointed; and on this upper platform the building with brick or stone may safely be commenced.

The same practice of laying the masonry with offsets as before described should be observed here; and although with piling and a planked bottom there does not exist the same necessity for building the first and second courses with large stones, yet the practice is invariably good, and ought always to be followed where such stones can be obtained without a great increase of expense.

Could any plan have been adopted for turning the course of the water, so as to lay dry the bed of the river, an engineer would be able, in the situation we have been describing, to build a bridge at much less cost than by the expensive means of a coffer-dam and piling. Where the river could be turned, the plan we should propose to give stability to the unsound bottom would be to cover it entirely over with cross sleepers of Memel logs, and on these to lay a covering of planks, closely jointed, while further security might be obtained by introducing inverted arches above the planking, between the piers, and extending under each of the abutments, as shown in fig. 8. It will however not be

FIG. 8.



necessary to introduce a platform of timber to support the inverts, unless the ground be too soft to construct them upon it, which however is often the case in situa-

tions similar to those we are describing; and in some cases the platform, without the inverted arches, may be sufficient of itself to carry the bridge: the determination however of introducing or dispensing with these must be left to the decision of the engineer when the lowest bottom can be seen; and all we can do at present is to point out the practice where difficulties exist.

It might be objected with reason to the above method, that the great weight of a bridge on such a foundation would cause it to sink to some extent, and if the bottom were very soft the settlement would no doubt be considerable. From this, however, there would be no danger of fracture, as the settlement would be gradual and uniform, and where the inverted arches are introduced they would most effectually prevent the piers and abutments being moved from their position. In support of this mode of foundation we may cite Mr. Rennie's example in dealing with a similar stratification, while building the Albion Mills close to Blackfriars Bridge. To avoid the expense of placing the foundations at a great depth, and to prevent settlement in the walls, he adopted the expedient of forming inverted arches over the whole space on which the building was to stand. To support the inverts where the ground was softest, some piles were driven, and courses of very large flat stones laid under the walls, while the inverted arches joining into these prevented any unequal settlement. Before quitting the consideration of unsafe ground, such as we have been describing, we must not omit to mention the use of concrete as an excellent

substitute for piling in every situation where this latter would otherwise be necessary to secure a solid foundation.

The expense of laying a foundation of concrete from four to six feet in depth, and extending about two feet round the space to be occupied by the building, ought in all such cases to be considered, and compared with the expense of piling, which will probably never be found more safe than a body of concrete, unless the bottom be so soft as to allow the concrete to sink into it, and thus entirely cease to support the building. For its durable and almost imperishable nature, concrete cannot be too much esteemed, and besides being quite as safe, and perhaps more durable than piling, its cost will in most districts be found much less. Where the piers are to be built either of brick or stone, a great saving will be effected by the use of concrete underground, as the expense of this substance will in no case exceed one-third the price of any description of brick work, and in some cases it may not cost more than a sixth. It may be thought by some that the description we have given is exaggerated as to the very unfavourable nature of the stratification in parts of the Crag district of Norfolk and the neighbouring counties; and in order to guard against this notion it may be mentioned, that in directing the works of the North Walsham and Dilham Canal in Norfolk, the writer had occasion to build a lock in a situation where, after sinking the lock pit down to the necessary depth for laying the sills at the lower gates, the bottom was so soft, that a couple of men,

without much labour, worked down an iron bar an inch square to a depth of twenty-eight feet. The bottom was then considered firm, but the expense of excavating to so great a depth through such a soft stratification, surrounded also by a mill-dam, was considered too great for the finances of the Company.

The following description of this work may be interesting, as an example of founding on a timber platform under circumstances of some difficulty.

The idea of introducing a platform of timber, supported by piles driven through the unsound strata to the solid ground, was also abandoned on account of the expense. The bottom was composed of peat or bog interspersed with sand holes, and some pits of fine gravel, which admitted water in all directions; under these circumstances the plan of foundation acted upon was that of laying transverse sleepers of Memel timber across the whole breadth of the lock, and under the side walls and counterforts. The clear width of the lock being fourteen feet, the side walls four feet six inches, and the counterforts four feet, it was necessary that these sleepers should be thirty-two feet long; they were one foot wide and six inches thick, laid at intervals of three feet apart, and extended from under the fore bay to twenty feet beyond the main sill of the lower gates. These sleepers were covered with three-inch planks, placed close together, forming a solid platform of sufficient size to support the whole lock, chamber, side walls, and counterforts, with an extension twenty-four feet further than the gates, in order to prevent the

water from undermining the wing walls when the sluices were opened in the lower gates. Sheeting piles were driven down at the front side of the mitre sills, another row at the front of the main sill, and a third row at the extremity of the platform.

These piles were four-inch beech planks, tongued and grooved, driven by a ringing engine, with a monkey weighing eight hundred pounds, and it is remarkable, notwithstanding the ease with which an iron bar was worked down to a depth of twenty-eight feet, that none of the piles could be forced down more than fifteen feet, and many of them were cut off at nine or ten feet. This difficulty was experienced in consequence of the quicksand adhering so firmly to the sides of the piles as to prevent them alike from sinking, and, when once driven, from being raised by any force that could be applied; so that if the workmen at any time after having driven a pile six or seven feet ceased their blows for only a few seconds, they were quite unable either to raise that pile out of the ground, or to drive it further in.

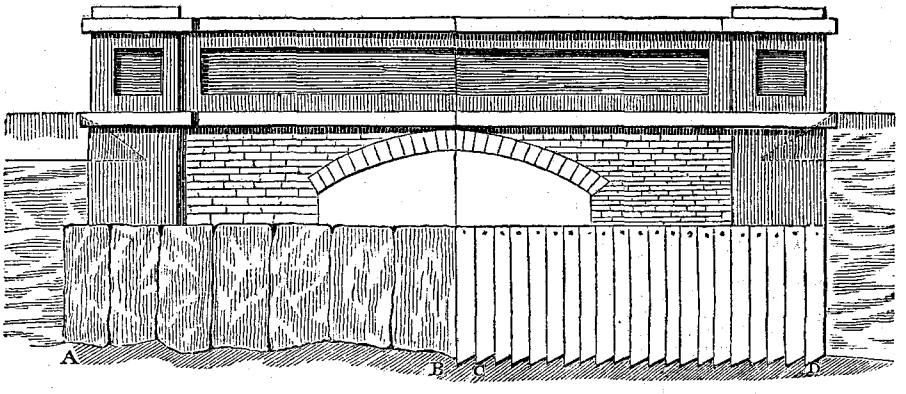
The uncertainty as to the nature of the ground in which piling is to be used has naturally induced engineers to specify indefinitely the length of the piles, because this can only be judiciously determined during the actual driving; and hence it has become a common practice to specify that the piles are to be driven down to solid ground, or otherwise it may be specified, as in the example we have given, that they are to be driven till fifty blows with a monkey of about eight hundred pounds weight will drive them no more than a quarter

of an inch into the ground. Piling is either paid for to the contractor entirely as an extra work, or he is directed to estimate for a specified amount of piling; and then he undertakes by his contract to execute either an additional or diminished quantity, as may be required by the engineer. The Company in the one case allow the value of the additional quantity, and in the other case deduct from the amount of his contract the value of the diminished quantity, if any. Mr. Telford in his practice as an engineer was exceedingly cautious, and never allowed any but his most experienced and confidential assistants to have any thing to do with exploring the foundations of any buildings he was about to erect. This scrutiny into the qualifications of those employed about the foundations extended to the subordinate overseers, and even to the workmen, insomuch that men whose general habits had before passed unnoticed, and whose characters had never been inquired into, did not escape Mr. Telford's observations when set to work in operations connected with the foundations. He was accustomed to examine men so employed whom he thought unsteady, and, if necessary, would reprimand the overseers for employing such men about the foundations in any capacity. It is evident from these precautions that Mr. Telford was well convinced how dangerous it was even to receive a report of the strata from men of careless habits or inefficient knowledge, and that he also knew the consequences which might follow from careless pile-driving, and, in short, from the absence

of proper care in all the operations connected with the commencement of an important structure.

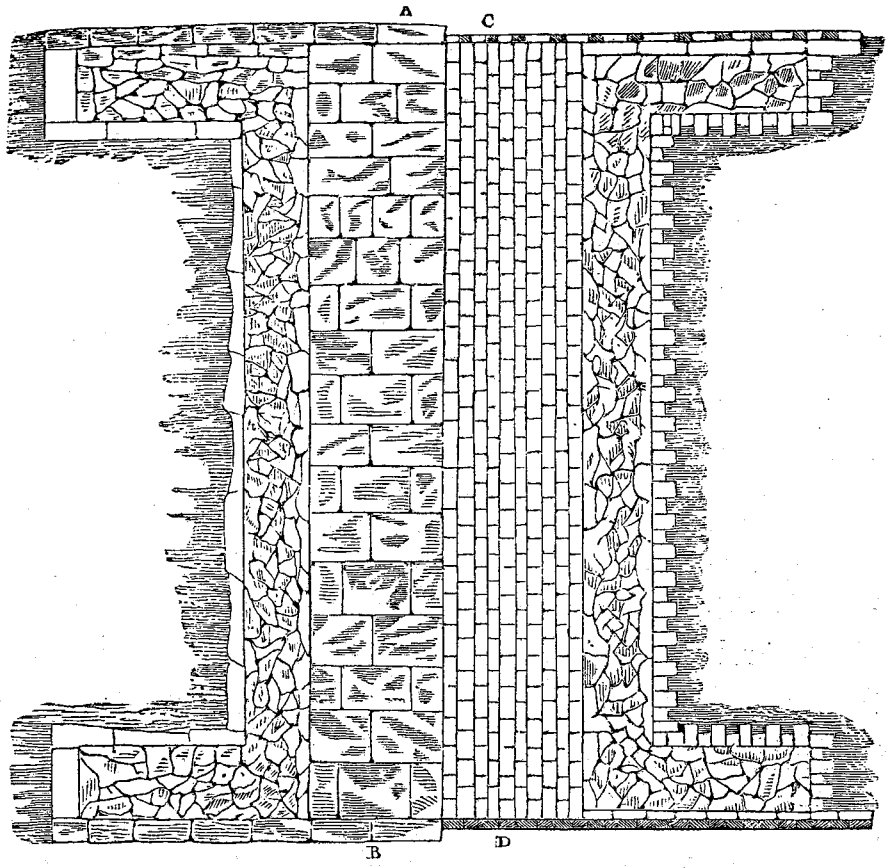
When the site of the foundations, on being explored, was found unsafe, and the expense of driving piles through a very deep bed of loose material would be considerable, Mr. Telford frequently laid an inverted arch between the piers and abutments, as already recommended and shown in fig. 8. At other places, where the ground was more solid, but still doubtful, a pavement of broad stones was placed over the whole bottom, extending also under the abutments and wing walls. These also were tied across the whole bottom by a row of the same description of broad-bedded stones placed on edge, and this kind of pitching was often continued to the extent of the wing walls, and then tied across by a row of the same stones, scabble-dressed, rough-squared, jointed, and set on edge; this plan is described in figs. 9 and 10: that half marked

Fig. 9.




AB in the plan and elevation is in reference to the practice we are now describing. Sometimes these stones could be procured in lengths of six or seven feet, from

FIG. 10.




three to five feet broad, and from six to nine inches in thickness; and when stones of these dimensions could be procured Mr. Telford invariably adopted them, as a covering, in preference to timber. In many places, where large stones could not so readily be obtained, he directed a pitched bottom to be formed with stones of about two feet superficial area, and not less than a foot deep. These were pitched endways, were close jointed throughout their whole depth, and always set with considerable care, the practice being never to permit them to be set on sand, or any material that could afterwards be carried away by the water. This pitching was usually

secured at the end of the abutments and wing walls by a longitudinal sleeper placed along the outside, with sheeting piles driven down in front to a depth of from six to ten feet, according to the nature of the bottom he was founding upon;—that part of figs. 9 and 10 marked CD shows this method. Amongst the variety of expedients adopted by Mr. Telford in his foundations of the Highland road-bridges, the one eventually decided on was usually undetermined until he became acquainted with the precise nature of the stratification; and even where he specified particular depths and forms of foundation, he always reserved the power of altering these in any way which after circumstances might lead him to decide upon. In his general specifications for Highland roads and bridges he directs, "Where there is no rock the foundation is to be sunk two feet below the bed of the river, and if the ground is soft, or the gravel loose, there must be a platform of timber laid under the masonry, to consist of two thicknesses of three-inch planks laid across each other, and if necessary have a row of pile-planking driven all round the outside." The practice here described was constantly followed, the rows of sheeting piles being always adopted if there was a current in the stream where the bridge was built; and if in consequence the slightest danger was apprehended of the bottom being wasted or carried away. Whatever might be the nature of the artificial bottom, this piling was never neglected where circumstances seemed to require it; and, as these will rarely be found precisely alike in any two situations, the expediency of piling must be regulated

according to the necessity of each particular case. In his general specification Mr. Telford goes on to say, " If the ground is hard, instead of a platform under the foundations there must be an inverted arch or pavement laid across, and wedged between the abutments, as wide as the bridge, well secured above and below by rows of stones sunk deep in the bed of the river." (This method of securing the inverted arches or pitching is shown in figs. 9 and 10.)

Such were his general directions with regard to foundations, the power of making alterations according to circumstances being reserved as before described. Connected with the subject of the Highland roads and bridges, Mr. Telford has left a most valuable scale of proportions, which he adopted in those works, and which will be found to combine elegance and beauty of design with strength and safety of construction. The proportions are given for all arches from four to fifty feet span, and as it will prove a most valuable guide to engineers, road surveyors, county surveyors, and all others who are concerned either in the repair or construction of bridges, we make no apology for inserting the Table in this place.

| Span of arches. | Rise of arch from the springing. | Depth of arch stones. | Height of abutment from the bed to the springing. | Thickness of the abutment walls on an average. | Length of parapets from the face of the abutments with wings under. | Height of parapets above the crown of the arch. | Thickness of spandrils and wings on an average. | Thickness of inverted arches, where necessary. |
|---|---|---|---|---|---|---|---|---|
| 4 | 1·6 | 1 | 2·6 | 1·6 | 9 | 1·2 | 1·6 | 0·9 |
| 6 | 2 | 1 | 2·6 | 2 | 10 | 2·2 | 1·6 | 1 |
| 8 | 3 | 1·2 | 2·6 | 2 | 12 | 3·2 | 2 | 1 |
| 10 | 3·6 | 1·3 | 3 | 2·6 | 12 | 3·2 | 2 | 1 |
| 12 | 4 | 1·4 | 3 | 3 | 14 | 3·2 | 2·6 | 1 |
| 18 | 6 | 1·6 | 3 | 4·6 | 18 | 3·2 | 2·9 | 1·4 |
| 24 | 8 | 1·9 | 4 | 5 | 24 | 4·2 | 2·9 | 1·4 |
| 30 | 12 | 2·0 | 4 | 5·6 | 30 | 4·2 | 3·0 | 1·6 |
| 50 | 15 | 2·6 | 6 | 6 | 36 | 4·8 | 3·6 | 1·6 |

derate velocity, as in some of the reaches in the Thames, the Crouch, the Blackwater, and other rivers on the eastern coast.

Sands of a sharper kind also are not unfrequently found in the form of quicksands, but when undisturbed by the motion of water are sufficiently solid to bear the weight of any structure. Experiments on the peculiar nature of quicksands may be made on any fine sandy beach, or on any tract of fine sand which is dry at low water. It will be found that almost as soon as the water sinks below its surface, carriages may be drawn over in safety almost without leaving an impression. But let any weight, having irregular surfaces, be placed on this kind of sand, and the projecting parts will sink into it; then cause this weight to be slightly moved by hand, or in any other manner, the sand will immediately give way, and, if mixed with water, be set in motion, and a real quicksand will be thus produced. It is not often found that clean dry sand, however coarse it may be in quality, will sink with the weight of a building; it may, however, be forced laterally out of its position, when the pressure is very considerable, and thus cause a sinking of the structure: therefore, when any danger of such a result is apprehended, the first part of the foundation should consist of a timber platform resting upon the sand.

The platform may be laid with sleepers ten inches wide and six inches thick, placed at the distance of three feet apart from centre to centre; these must be closely covered with four-inch thick planks, close jointed and secured to the sleepers with trenails. This platform must be of

sufficient superficial area to extend two or even three feet on each side beyond the base of the structure raised upon it. The exact superficial area, however, of the extension must be regulated by the height and solidity of the building, and the possibility of lateral space existing into which the sand foundation might be forced; but in any case where a building has to be erected on a dry sand foundation, it may be made perfectly safe and unyielding to any weight that can be placed upon it by adopting the platform covering, as described, and of sufficient superficial area, the dimensions of which area must be determined according to the circumstances above stated.

In England, where the sand, whether coarse or fine, is usually composed of hard silicious particles, very little danger of settlement need be feared; but where, as in the bed of the Seine, the sand or small gravel consists chiefly of rounded pieces of chalk without much admixture with any binding material, it is extremely probable that settlements would occur, as the effect of great pressure would be to crush and break the softer particles, and thus cause the under stratum to occupy less space than before the interstices were filled.

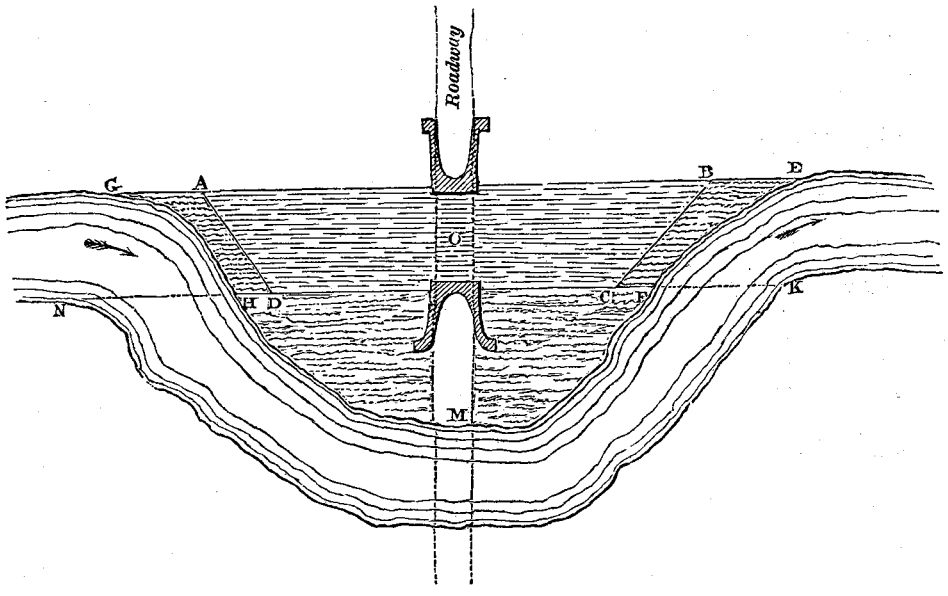
It may be concluded that almost every description of sand might be safely built upon if it could be protected from the action of water, which, when set in motion, is quite capable of carrying away such material, and of pressing it from under any superstructure which may rest upon it. In the peculiar case of bridge foundations, however, it is very seldom that a sand stratum so safely

circumstanced can be met with, because, being usually in the neighbourhood of large bodies of water, and the sand being of a porous nature, water will always be found filtering through, so as to render it an unsafe bed to build upon. Hence it is scarcely possible to find a bridge foundation established upon a sand bottom without the accompaniment of a platform, generally laid on piles. Notwithstanding all this, there are situations in which the practice of dispensing with timber on a sand bottom would be perfectly safe. For instance, where there is very little current in the river and the sand bottom has a covering of from two to three feet of clay or heavy gravel, which is never liable to be disturbed by floods or other causes, a building might safely rest on a close sand bottom. In other cases, an artificial covering may be placed over the sand, and thus prevent its disturbance. Such a covering also will often be found necessary after the erection of a bridge, in order to prevent a sandy or silty bottom from being carried away by the current; although before the erection of the bridge the current might not have had sufficient power to set the sand in motion. An instance of this kind occurred in the case of the Lary Bridge, at Plymouth, where the current being increased in velocity by the contraction of the waterway was found to act on the bottom in such a manner as to endanger the foundations; and in order to prevent serious consequences, Mr. Rennie adopted a covering of clay in the bottom, overlaid with stones varying in weight from two hundred pounds downwards, the whole thickness not exceeding two feet. This cover-

ing, which extended a distance of sixty or seventy feet above and below the bridge, became very solid, and effectually prevented the further wasting of the bottom.

When, instead of building a bridge in the existing channel of the river, it is considered eligible to build on the dry land and direct its course through a new channel,

FIG. 11.




as shown in the accompanying figure, it would be adviseable to proceed somewhat in the following manner.

The first operation to be performed is that of excavating the new river-course marked in the figure GEFH, leaving, however, a solid portion at each end of sufficient strength to prevent the water breaking through, and to admit a temporary road for the traffic during the time of excavating the new river-bed and abutment foundations, as represented in the figure by the letters GADH at the upper end, and the letters BEFC at the lower end, of

the stream. The earth taken out of the new channel may be laid in the seat of the new approach, or, if more convenient, on the small insular piece of land between the old and new channels marked HMF, from which place it may be removed into the old river-channel when the water is turned the other way, or it may be employed in completing the new approach.

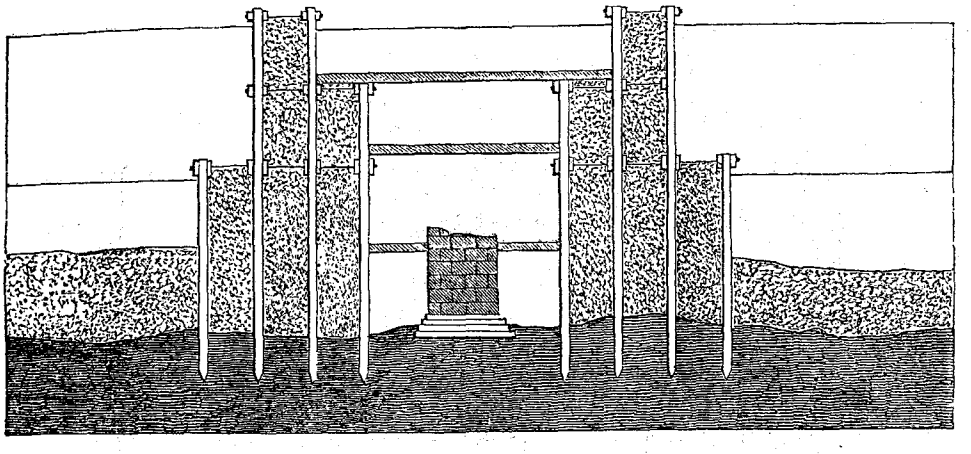
The engineer will decide upon the disposal of the excavated material according to the facilities of obtaining earth within a reasonable distance for making up the approach, in case that which is taken from the new channel should be used for filling up the old river-bed. As soon as the bridge has been built on the new cut in the situation marked O, the stank, or portion left at the lower end of the new cut, must be taken out, and afterwards the portion of earth marked GADH at the upper end must also be removed, and run by barrows into the old river-channel in the direction NH, in order to turn the water into the new course. Should the situation where this work is required be in retentive ground, which will not admit of water passing through it, the operation of turning the river may be very simple and attended with little difficulty. If, however, the stratification is of an open porous nature, as soon as the excavation is carried down as low as the water in the old channel, springs will be found to rise in the new excavation, and unless it can be drained by bringing up a level from the lower end of the cut at F, no better method remains of getting out the bottom than that of dredging with bag and spoon, or with a steam dredging engine.

It is necessary to observe, with reference to the expedient of turning a river in order to build on dry land, that the whole operation will be found very troublesome and expensive when the soil is porous and incapable of retaining water. In addition to the cost of getting out the new excavations under water, it is by no means unlikely that the same expensive foundation of piling and timber platform will be as indispensable now as if the bridge had to be built over the old channel; and the newly opened ground may, by the action of the water, be rendered so porous as to permit water to enter the openings for the piers and abutments to an extent requiring, before the building can be properly carried on, the construction of coffer-dams almost as expensive and formidable as if they had been made at first in the middle of the river. Hence it is necessary to use considerable caution in deciding upon opening new ground to build the bridge upon. Pits ought, if possible, to be sunk; indeed, they are absolutely necessary in all cases of large bridges; they alone will afford the engineer the proper means of comparing the different soils he has to contend with, both in the new and the old channels, and his judgment should of course be formed only after a due consideration of these and other particulars which he will acquire in the course of his investigations. Borings are not to be depended upon, and where the stratification is at all doubtful they ought never to be attempted. Trial-pits should be sunk as deep as to satisfy every inquiry.

In pursuance of the intention formerly expressed, we proceed to give some details for the construction of a coffer-dam. By way of example, suppose it has been determined to place a bridge pier in a tide river, where there is ten feet depth of water at the lowest spring tides. The bottom has been found to consist of twelve feet loose gravel and sand, with clay underneath. In this situation suppose the depth at high water to be not less than twenty-eight feet, making the whole depth, from surface of high water through the loose bottom down to the clay, forty feet. It will be necessary, in order to form the coffer-dam under these circumstances, to drive four rows of piles all round the space to be built upon. The clay puddle will then occupy three distinct spaces, called puddle walls, between the four rows of piles. The lengths of the piles may be as follows:—outer row to be driven down to within a foot of low water mark, and five feet into the clay, making their length twenty-eight feet; the two middle rows to be also driven five feet into the bottom, and to stand three feet above high water mark, making their length forty-eight feet; the inner row of piles to be driven to about eleven feet above low water mark, and five feet into the clay, so that their length will be thirty-eight feet. The plan and section of this dam, fig. 12, will show the form of construction, and the following short outline of a specification will render the whole intelligible. The clear breadth between each two rows of piles to be six feet; the outer row of dam piles to be half logs of twelve inches by six inches

TRANSVERSE SECTION, FROM A TO B.

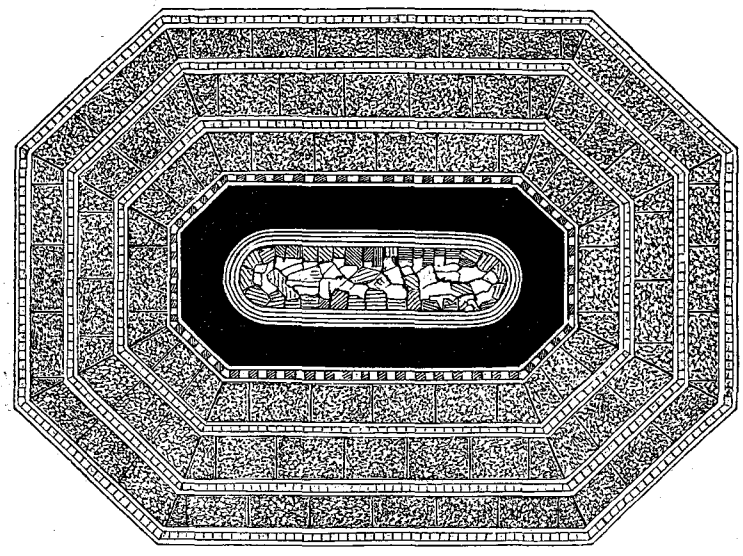
FIG. 12.



scantling; the middle rows of piles to be twelve-inch square logs; and the inner row of piles to be twelve by eight inches scantling. The whole of the timber for the piles to be straight grown, and to consist of the best Memel logs. A double row of waling-pieces to be placed all round the top of the inner piles, as shown in fig. 12, and to be connected by wrought iron bolts an inch and a quarter square. The bolts must have good heads, one inch thick and three inches square, and the other end must have a fine screw-thread cut upon it, with a three-inch square nut, one inch thick, made to screw on at the end of each bolt. Also between the timber and each of the nuts there must be a plate or disk of iron, called a washer, to prevent the nuts from pressing into the timber when tightly screwed up, and also to prevent the friction between the timber and the nut, which would be very considerable if the latter were screwed in contact with the wood. In the same manner several other double lines of waling must be fixed round

PLAN OF COFFER-DAM.

FIG. 13.



the whole coffer-dam, precisely as shown in the transverse section, fig. 12. The whole of the waling to be of the same quality as the timber for the piles, and to be of scantling equal to one foot by six inches.

Connecting bolts must be introduced at intervals of four feet from bolt to bolt all round the dam, as shown in the figures.

In the situation we have supposed for the construction of this dam, where a stratum twelve feet in depth of sand or gravel rests upon the clay, it will be necessary to excavate the whole of the former, which, on account of its porous nature, would otherwise permit the water to penetrate through it into the dam. This will occur however perfect and water-tight the clay filling placed above the porous stratum may be, because the water will pass under the water-tight material, and fill the interior

up to the level of the water outside. Hence the necessity not only for driving the piles through such a stratum as this, but even of entirely removing it, in order that the artificial filling-in between the piles may commence by a perfect junction with the clay beneath, and be carried up tight to the necessary height. The whole of the porous material therefore must be removed from the space to be occupied by the dam; and it will be found most convenient to effect this before driving the piles, because there will then be no obstruction to the working of a dredging engine. With respect to the filling-in between the piles, it is very important to distinguish with accuracy the material which is proper for this purpose from other kinds of earth, which, although known by the same general name, clay, would yet be found quite inefficient for the puddle of a coffer-dam. In specifications for these works, the usual expression is that good retentive clay must be filled in between the rows of piles, and on the judicious performance of this very indefinite injunction depends the tightness of the dam. Considering only the two extremes of very hard and very soft plastic clay, it will be found that the former of these, when broken up and thrown in between the piles, will seldom or never form a perfect dam. On the contrary, vacuities will remain between the broken pieces, and it will be found exceedingly difficult to beat down clay of this kind into a body sufficiently firm, compact, and solid to resist the efforts of the water to penetrate through it. If, again, clay of a very soft plastic nature be introduced, it will partially dissolve and com-

bine with the water when thrown into it, so that the space between the piles will be filled with a kind of mud puddle almost in a fluid state, of no greater consistency and no greater capability of keeping out water than mud itself. It is evident therefore that either kind of clay by itself would not answer the purpose intended of forming a solid water-tight puddle. All the clays, when used in a coffer-dam, require a mixture of gravel and sand, or a portion of pounded chalk will be found an excellent material to give solidity to the soft portion of the clay, and to fill the vacuities and interstices which may be expected to exist where the clay is of a hard and lumpy description. However general may be the opinion, it is certain that one more erroneous was never entertained than that clay alone is a proper material to make a good puddle-dam. Clay by itself is subject to great changes, according to the alternations of heat and cold, drought and moisture. In very dry weather, and when exposed for a time to the influence of the sun, all moisture will be extracted; and the clay will invariably crack and separate into a number of irregular fragments, which will never afterwards unite so as to form an adhesive water-tight substance. The difficulty of compressing clay, when placed in a dam of any considerable depth, into a solid mass without hollows has been already noticed. If in addition to this objection we consider the immense weight and pressure of clay so compressed against the piles forming the sides of the dam, and the consequent strain on the piles, which ought only to be employed in resisting the pressure

of the water from without, we shall see sufficient reason to decide, on these as well as on other grounds, against the practice of puddling entirely with clay. From the very best information which can be brought to bear on this subject, namely, that derived from long and watchful experience, accompanied by the knowledge that he has himself, as a contractor, lost large sums of money on account of too great a faith in clay puddles, the writer is enabled to speak very positively on the nature of this material, and in addition to the objections already advanced begs to add his own personal observations of the fact that puddles composed entirely of clay have usually bulged, given way, and been found incapable of keeping out the water when of considerable depth, and that in any case a puddle with an admixture of gravel, chalk, and sand will make a safer water-tight dam than clay alone. In order to secure a proper construction of coffer-dam, we recommend that all specifications should very fully particularize the manner of filling the coffer spaces; thus, the kind of clay to be used,—the quantities in which it is to be thrown into the dam,—the size of the pieces into which, if hard clay, it must be chopped,—and the mixture which is to be added, whether of sand, gravel, loam, mould, or chalk,—should all form particular instructions in the specification. It is certain that in great depths of water the practice ought never to be dispensed with of mixing some foreign material with the clay, otherwise it will not set or become hard under water. We have mentioned sand, gravel, and chalk, as proper materials for this purpose, but it is impos-

sible to assign any proportions to be adopted in the mixing, because these can only be determined as a matter of judgment on seeing the kind of clay to be used, knowing its tenacity and capability of resisting pressure, and also whether in the state it is dug it contains already any mixture of gravel or other foreign ingredient. Probably three parts of pure clay, two of chalk, and one of fine gravel, would make an excellent compound for filling the dam. These materials, however, should be well mixed together, the chalk and clay chopped small, and no stone larger than a hen's egg allowed to pass amongst the gravel.

It is usual in large dams to cover the top of the puddle with bricks, but a foot in depth of good strong gravel, grouted with lime, is much cheaper, and answers equally well with the covering of bricks.

All the piles used for the dam must be shod with good sound wrought iron shoes, weighing not less than 10 lbs. each, and hooped also with iron of the same kind by rings three inches broad and three quarters of an inch thick, to prevent the timber from splintering or otherwise giving way under the driving.

Having dwelt thus at large on the form of construction, and described in detail the component parts necessary for a first-rate coffer-dam, it might be thought by some that the following brief remarks on the means of carrying into effect the actual working operations of forming a coffer-dam are too exclusively within the contractor's province to be of any use to the engineer. But, on the other hand, considering how great is the difficulty

of drawing a line to separate the duty and responsibility of the engineer who directs from that of the contractor who executes, and, further, admitting that the superintendent of a work cannot be too intimately acquainted with the details of execution even independently of the materials to be dealt with, some such information, with reference to coffer-dams, can hardly be thought uninteresting to the engineer.

The variation of circumstances under which coffer-dams are constructed will occasion differences in the facilities, and generally in the entire mode, of execution, because the same means which in one case may be successfully adopted for driving the piles will be quite inapplicable in other situations where the difficulties are perhaps much greater. For example, in many rivers and canals it might be possible, without obstructing the navigation, to erect a temporary fixed stage, on which the pile engine could stand, and proceed without interruption in the work of driving the piles. But in a tide river it will scarcely be possible to drive from a fixed stage, and it is therefore necessary to employ barges or other floating vessels on which the pile engines must be erected. The barges used for this purpose are generally of forty or fifty tons burden, and by proper management of these the pile engines may be made to work at all times of tide.

The first operation in constructing a coffer-dam should be that of driving the guide piles, which are so called, because, being the first which are driven in each row, they serve as a guage and a guide both in position and

scantling for the rest of the piles. These guide or guage piles are usually placed about ten feet apart, and as they are driven down without any marks to indicate their proper position, except the known direction of the sides and ends of the dam, and of course are driven before any waling can be fixed to preserve them in a perpendicular direction, the business of driving them is one of more difficulty and complication than that of inserting the intermediate piles.

In order to drive these guide piles, the vessel containing the pile engine should be moored stem and stern alongside the line of the intended dam, so that as many as possible of the piles can be driven without altering the position of the vessel. In this first part of the process a ringing engine and wooden monkey of about eight hundred pounds weight will act with more success than a heavy iron ram, and when it is found that the piles will not go deeper without skewing from the perpendicular direction it is better to discontinue driving them, and leave them to be forced down afterwards when the others have been driven. As soon as all the guage piles of any one row have been fixed, the walings should be fastened to them, and the intermediate piles, ten in number between each pair of guage piles, may be driven down. It is scarcely necessary to say that the waling will be of important service in driving these piles, because it will determine their precise position, and serve to keep them upright during the process. Considering the great inconvenience and delay in the execution of a work which may arise from an imperfectly constructed coffer-dam,

we would recommend that the engineer should always direct the contractor in his specifications to place all the piles intended to be used in some convenient spot for inspection before they can be admitted into the dam. The piles may then be laid close to each other, their scantling, and particularly their breadths, properly measured, and their angles tried in order to see that they will fit close to each other, and fully occupy the space between the walings. This preliminary fixing is just as necessary as that adopted in trying roofs and centres before they are actually erected, and if insisted on by engineers, in the case of coffer-dams, would prevent many evil consequences which may be traced entirely to the use of improper piling in the dams. It is certain that if piles laid down on the ground side by side before being driven will not fit close to each other, they never will afterwards, and under such circumstances it will ever be found exceedingly dangerous to try them.

When the engineer is satisfied that the piles are properly prepared, and can safely be used in the work, the whole number between two of the guage piles should at once be placed upright for driving, instead of being inserted and driven down singly. This method will in a great measure prevent the piles from taking a sloping direction, and will ensure the desired result of properly filling up the spaces between the guage piles with a close-fitting compact row of intermediate piles. We should recommend that the driving be commenced and continued as long as practicable with a well-constructed ringing engine, and that when the piles cease to sink

fairly after repeated blows from the monkey, a heavy iron ram, with considerable fall and weighing fifteen or eighteen hundred pounds, be substituted to complete the driving. Care should be taken not to allow the heads of the piles to get bushy or besomy; and when this appearance is observed, the heads should be immediately squared off, in order that the force of the blow may not be deadened, but may produce its full effect by falling on a solid substance. If this precaution be neglected, the driving cannot be done without great difficulty.

Before deciding on the particular kind of foundation which should be adopted under any given circumstances, the engineer ought to inform himself with every possible care as to the nature of the bottom, not only as he may find it at the time of his examination, but also with reference to the possibility of future changes, which might entirely alter its character.

The necessity for such an extended consideration will be evident from an observation of the effects produced by water on all stratifications with which this powerful agent is found in contact.

We have seen that sandy soils, when saturated with water, become quicksands; and if we extend our notice on this subject to other soils, we shall find changes of structure no less remarkable in clay, chalk, peat, &c., according as these are saturated in a greater or less degree. The difference between hard dry compact clays and the softer kinds, which may be dug with an underhand grafting tool, is sufficiently remarkable. With respect to chalk, we may distinguish that upon which

water has extensively operated since its formation as the substance known by the name of chalk-marl, a material widely different from the dry beds of hard flinty chalk which more commonly prevail. The effect of water upon peat also may be considered identical with the production of ooze, which is nothing more than peat in which the vegetable structure is entirely destroyed and the peat converted into a soft and very compressible substance, sometimes of a consistency not much harder than butter. Now, as it is a fact of very common observation that great changes take place in the circumstances under which water is found pervading various strata, it becomes obviously of importance to investigate thoroughly all these circumstances before deciding upon the method of constructing an important foundation. This remark demands particular attention whenever it is in contemplation to introduce a timber foundation, because, independent of the settlement or otherwise giving-way of the stratification, which the changes we have alluded to may occasion, it is well known that the change from a state of moisture to that of drought produces the most injurious effects upon all timber. Thus it is found that timber foundations originally laid down in wet marshy districts, which have afterwards been drained, have decayed and endangered the stability of the superstructure; the same result has followed in many situations also where the water has receded from natural causes, or where wet and dry seasons affect the moisture of the ground. In the neighbourhood of large towns the sinking of

numerous wells will frequently withdraw water from a timber foundation and thus cause its decay, while, if the original state of moisture had not been disturbed, the foundation would have long remained perfectly sound. Even the temporary variations from wet to dry, occasioned by the ebb and flow of the tide, are found very injurious to piling. The decay of timber foundations in the situations we have been describing is made apparent in various ways, but usually by cracks or fissures from top to bottom of the building, occasioned by the want of support in those places where the timber has decayed. These cracks will be more or less extensive according to the height and weight of the building, but in any case are dangerous, to say nothing of the disfigurement they produce. In order to guard against this decay of the timber, the foundation ought, if possible, to be laid low enough to ensure the object of keeping the wood-work always moist, wherever any danger exists of the water being withdrawn. To effect this, however, would often occasion a vast expense, and it will then be found most expedient to abandon the idea of using timber in the foundations.

Before quitting the subject of piling, we feel it necessary to advert to a most injudicious practice which we have seen adopted in many situations where the bottom is considered unsound. We allude to the system of partial piling, which is frequently used in parts of a foundation, while in other places of the same foundation the ground is considered sufficiently hard to bear the necessary weight without the security of piles. The

obvious effect of this system is to produce in the foundation certain very hard points of support, while the remaining surface of the bottom, namely, where no piles have been driven, is usually compressible, and unable to bear the great weight upon it without yielding in some slight degree. It must be observed that settlement or sinking will occur in almost all buildings, but provided this be perfectly uniform in all parts of the foundation the consequence to the building is by no means injurious. But conceive one part of a foundation yielding even to an extent ever so trifling, while other contiguous parts remain perfectly solid and immoveable, and we may readily arrive at one very common cause of settlement in buildings. The part of the foundation which sinks under the pressure of the building may be considered as a hollow lying under the stone-work, while the effect produced upon each course of stones may be compared to the strain upon a beam projecting from a wall and loaded uniformly with a weight. Each course of stones is in fact a beam, supported at one part on the hard piled foundation, while the weight of the building above, tending to press the other part down into the yielding part of the foundation, acts with great leverage, and usually produces a fracture.

To investigate the direct tendency to fracture occasioned by a partially unsound bottom, and thus show in the clearest manner the danger of allowing so great a weight as that which usually presses on the piers of a bridge to act in such a destructive manner, would by no means be difficult; but as this would be more in the

nature of a theoretical inquiry than the object of this paper will justify, it may be sufficient to point out the manner in which this kind of destruction is caused.

At present the practice of partial piling is notoriously common, particularly in the neighbourhood of the metropolis and other large towns, where the builders not unfrequently take great praise to themselves for the judgment they exercise in order to determine what parts of the foundation for walls, &c., should be piled, and what parts may safely be built upon without piles. Such an injudicious system cannot be too strongly condemned, and engineers and architects ought to be very careful how they trust to the management of contracting builders, and others having charge under them of extensive buildings. In order to show the consequences of even a very slight partial settlement in a building, as illustrating the danger of adopting a system which is actually calculated to produce such a settlement, we shall relate an instance with which the writer himself was intimately acquainted. The piers of a large aqueduct, eleven in number, with two abutments, had all been founded on gravel, a few feet below the surface, and stood remarkably well, the masonry appearing without a flaw when they were carried up to their full height of about fifty feet. One of the piers at the south end, however, was founded one part on the gravel and the other on very hard Whinstone rock, the surface of which was merely levelled and the building at once commenced. When carried up to about thirty feet a formidable fissure was observed from top to bottom of this pier, and the

only possible source to which the mischief could be traced was the step of founding the pier partly on the rock and partly on the gravel. Had the whole pier been on the rock, it would of course have stood without any settlement; had the whole been on the gravel, it would perhaps have settled to a trifling extent, but would no doubt have stood as well as all the other piers, which were founded entirely on the gravel. Placed, however, partly on the rock, which was perfectly solid, and partly on the gravel, which slightly yielded beneath the great pressure upon it, the consequence followed as described above; in order to remedy which, a great number of the stones had to be taken out and re-set on each side of the fissure. The application of this example, which shows the consequences of building on a natural foundation more solid in one part than another, to the case of an artificial foundation put in precisely the same situation by the injudicious use of piling, is sufficiently obvious. In defence of the practice of partial piling it has been urged that the ground in its original state is already more solid in one place than another, and that the piling is only introduced to render the whole surface equally firm. But the answer to this obviously is, that the introduction of one evil is no cure for another, and it would probably be found better to build on the original ground, bad as it is, than to adopt a foundation so unequally sound as that which is formed by the use of piling in occasional spots. Where the unsound parts of the foundation are of small size, a few flat-bedded stones placed in the bottom will frequently answer as well as

piling; but if unsound to any considerable extent, the engineer will prefer the use either of piling or of concrete all over the foundations to the imperfect and unsafe expedient of patching up the unsound parts with piles. Looking at the usual effects of partial piling in a suspected foundation, it would seem to be really worse than useless, and the obvious rule pointed out by experience appears to be that piling, whenever adopted, should extend entirely over the whole base of the foundation, and even some few feet beyond it all round. The piles should be driven at equal spaces apart, and regularly planked over, as described in a former paper.

Some French engineers practise a system of rendering the ground to be built upon uniformly solid by means of pounding with a heavy rammer, and wherever, by this process, any part is beaten down below the general level, the space is filled up with other material, until the whole is well compressed and brought to a level surface. With some kinds of clay this system of pounding may answer very well, but in other stratifications it would be rather injurious than otherwise. For instance, where intermixed beds of clay and sand form the stratification it is frequently found that some parts are less solid than others, and yet the practice of pounding would probably produce no good effect upon the defective parts, as it would be very apt to occasion water to spring up from the porous strata beneath. Thus in building on almost any part of the London clay, which contains numerous beds of sand, it would by no means be judicious to try the expedient of pounding. We are inclined to think

that dry clays, mould, and newly made ground, present the only cases where pounding would be of any advantage in effecting the object of rendering the ground uniformly solid.

## No. 4.

THE value of concrete, as a substitute for stone or timber in foundations, is now well known, and the situations will be found exceedingly rare where this substance cannot be successfully employed. If tried, however, on a quicksand where the water could carry away the particles of sand, or on gravel where the finer particles could be similarly acted on by water, and the whole stratum thus lowered, hollows would be formed under the concrete which would in all probability give way and much endanger the stability of any building resting upon it. In a situation of this kind piling would be infinitely preferable to concrete, and in like manner it would be better to use piles than to lay concrete on any soft stratum into which the concrete would sink. The cheapness of concrete is one great recommendation which ought never to be overlooked. When an unsound bottom has been tried, and the fact ascertained that its depth is inconsiderable, say five or six feet deep, it would be on the one hand very bad engineering to attempt building with stone upon the unsound stratum itself, and on the other hand, on account of the expense, the error in judgment would scarcely be more justifiable of taking out the unsound stratum and building with stone from the solid; but this is a case where concrete may be used with great advan-

tage, because it may without danger be placed on the unsound stratum; and in a much greater degree will concrete foundations be superior to all others where the depth of the unsound strata is considerable, and where, except for the concrete, the extensive use of piling would be absolutely required.

Objections have been urged against the use of concrete in marshy grounds, and supported by allusion to the Greenwich Railway viaduct, the piers of which were founded on concrete, and some considerable settlement afterwards took place. The writer is not sufficiently acquainted with the details of that work to be able satisfactorily to explain the causes of the settlement, but is certainly not inclined to attribute any evil consequences to the use of the concrete in that situation, because he is persuaded that this material is the best that could have been adopted for a foundation in those marshes. This opinion is corroborated by the fact that the Croydon Railway viaduct, which joins the Greenwich Railway about a mile and a half from London Bridge, and is founded in the same marshes, even in the very worst part of them, stands remarkably well, and presents no trace of any flaw, crack, or settlement.

Numerous examples might be pointed out where buildings of great extent are found to stand remarkably well on concrete foundations, placed, too, in a stratification far from solid. Amongst these may be mentioned the Penitentiary at Mill Bank, and the very large building called Fishmongers' Hall, on the north side of London Bridge. Both these buildings rest on a

substratum of very soft elastic alluvial soil; and yet their foundations, of concrete, are unrivalled for firmness and stability. The Penitentiary, particularly, is an old building, and may be cited as one which has been fairly tried by the test of time. Pieces of concrete which have occasionally been taken from the foundation of this building are not inferior in hardness to some of the pudding-stone rocks found in the neighbourhood of the coal fields.

The kind of limestone from which the concrete is made very importantly affects its quality. The following are the principal varieties of strata from which limes are burnt in this country.

1. The upper and lower chalk, extending over a considerable portion of Lincolnshire and Yorkshire, ranging also through the counties of Norfolk, Suffolk, Hertford, Bucks, Wilts, Dorset, Hants, Surrey, Sussex, and Kent. 2. The Ashburnham limestone, deriving its name from the locality where found in the county of Kent. 3. The oolite formation, in which are numerous beds of limestone, throughout the counties of Dorset, Somerset, Gloucester, Oxford, Northampton, Rutland, Lincoln, and York. 4. The lias formation, extending from Lyme Regis, in Dorsetshire, through the counties of Dorset, Somerset, Gloucester, Warwick, Leicester, Nottingham, Lincoln, and York. 5. The magnesian limestone formation, extending over the eastern parts of the county of Durham. 6. The carboniferous or mountain limestone, extensively prevailing in the north of England, in Cumberland, Northumberland, Yorkshire, Derby-

shire, also in Denbighshire, Monmouthshire, Glamorganshire, and Somersetshire; and, lastly, the Silurian rocks, in which are numerous beds of limestone, particularly at Wenlock, Dudley, Llandovery in Carmarthenshire, &c. It is true that limestones are found in other formations besides the preceding, for instance, occasionally in the old red sandstone, and more rarely beds of limestone occur in the coal formation. We have, however, enumerated all the rocks from which limestones are usually quarried for the purpose of being burnt into lime. The qualities of numerous limestones appear to have been well considered by Colonel Pasley, and of all those mentioned above he gives a decided preference to the lias limestones, when speaking of the best description of lime for concrete. This author condemns the use of the pure limes, which do not possess the property of setting readily under water, because, in such large masses as are requisite for the practical purposes of forming artificial foundations under heavy buildings, the concrete made of such lime would never have time to set in a damp situation, and wet would destroy it. The limes of Dorking, Merstham, Reigate, and Halling, however, which are all procured from the beds of the lower chalk formation, usually called the grey chalk, possess hydraulic properties in a greater or less degree, and will be found an excellent material for concrete. All of these places supply the metropolis with lime, which is, no doubt, extensively used for concrete. In the application, however, of lime burnt from chalk, care should be taken never to confound the upper with

the lower chalk, because none of the beds containing flint possess the qualities of water limes. In fact, the lime from the upper chalk usually makes weak mortar, and therefore is quite unfit for concrete. Water limes are procured at several places in the range of the Sussex South Downs, as at Poynings, Clayton, and elsewhere in the neighbourhood of Brighton. It is usually considered that the magnesian limestones possess hydraulic properties, but, as an ingredient in concrete, they are certainly inferior both to the lias and to the stronger of the grey chalk limes.

It seems that all the water limes of this country contain a certain portion of clay, and of this ingredient the analyses of some of the magnesian limestones show as much as eleven per cent. The carboniferous or mountain limestones, abounding so extensively in the neighbourhood of the coal and iron districts of Great Britain, burn into a very strong and excellent lime. It must, however, be observed, that limestones generally require time and fuel for burning in proportion to the strength of the lime they yield; and this, again, in limestones of the same composition, is usually in proportion to their hardness. The chalk seems to be an exception to this rule, which certainly applies to all other limestones with which we are acquainted. Probably the most easily burnt lime that can be found is made from the chalk marl of Norfolk, but this lime, although very pure in quality, is exceedingly weak. The ordinary dry chalk also burns with great ease, and the lime may be considered of medium quality; and, lastly, the grey chalk

burns with very little more difficulty, and the lime produced is excellent. Many valuable beds of limestone are met with in the oolitic series, particularly in the lower or great oolite, where those beds are frequently found resting on the white or yellowish-coloured freestone, from which, more particularly, the oolite formation derives its name. The oolite limestones most proper for burning into lime are mostly of a dark mottled grey colour, not unlike the lias; they burn with difficulty, but the lime is of excellent quality, and much valued for its strength and durability. The writer thinks it probable that the oolitic limestones in general possess hydraulic properties, but cannot speak positively on the subject.

The Silurian rocks furnish a great variety of limestones, some of which are so very impure, and so filled with large fossils, which have introduced animal matter into combination with the calcareous basis, as to render them quite unfit for being burnt into lime. In the neighbourhood of Dudley, however, and in other parts of the Silurian range, considerable quantities of lime are burnt. Considering the usually exposed situation of concrete foundations in regard to the action of water, and the consequent necessity that the lime used in its formation should possess to a considerable degree the properties of setting and remaining hard under water, it is always advisable, if possible, to use a strong lime, if not a regular water lime, as the lias. The mountain limestones burn into a lime which sets remarkably hard, and may safely be depended on in all concrete founda-

tions; on the contrary, none of the lime made from chalk marl, or dry chalk with flints, ought ever to be used in wet situations, since it will never set or become hard when continually subjected to water in contact with it. Before determining on the proportion of lime that should be used in making concrete, the quality and size of the gravel ought to be considered, because it must be observed that the quantity of lime should be increased according to the coarseness of the sand used with the gravel.

It is evident that in the composition of concrete the sand or the finer particles of gravel combine with the lime as in ordinary mortar, and it seems reasonable that the quantities of quicklime and sand should bear the same proportion to each other in good concrete as in good mortar, because the obvious condition sought to be fulfilled is that the pebbles of the larger gravel shall be cemented together in the most perfect manner. Again, as the fluid mass of water, lime, and sand must fill the entire of the interstices between the larger particles, the quantity of this mass must vary with the nature and size of the pebbles. It is not true that the larger pebbles placed in mass always contain more unoccupied space than the smaller pebbles, this being a law which only holds where the stones, great and small, are to a certain extent homologous and similar figures. It is clear, if the smaller pebbles are mostly rounded, and the larger mostly angular, the former may very readily contain more space or vacuity than the latter, the masses of each being equal. As a general rule, the gravel used for concrete should not be too fine, and, at the same

time, large stones are inadmissible. The bed of the river Thames in many places furnishes an excellent mixture of sand and gravel for making concrete, without any preparation whatever. Concrete used in the foundation of the Penitentiary was made of this Thames gravel, and certainly a better specimen of an artificial foundation can nowhere be met with. Colonel Pasley relates the fact that part of the foundation of the Penitentiary was found to be giving way; and the writer of this article is able, from an acquaintance with the history of this first use of concrete, to account for the giving way alluded to. Mr. John Hughes, the father of the writer, happened to be the contractor who executed the concrete foundation. In consequence of the interference of some of the inspectors or superintendents, Mr. Hughes was compelled, contrary to his own remonstrances and strong objections, to screen the gravel, and thus separate all the sand from it previous to mixing it with the lime to form the concrete. The consequence was one which might with great certainty have been anticipated; the lime mixed with the large gravel alone, without sand, even if it hardened at all, instead of setting into an artificial stone had no more strength than dried clay, and none of the solidity and strength of good concrete or artificial rock. Hence the reason of the failure, which was prevented from proceeding to a greater extent by a subsequent order to use the gravel as it was raised by the engine out of the bed of the river, without any preparation whatever. It follows from what has been said already of the composition of concrete, that the same quantity of pebbles in pro-

portion to the lime and sand must be used, whatever be the quality of these, and hence the subject which requires the most important exercise of judgment is that of determining the relative portions of lime and sand to be employed as the cementing ingredients, or matrix, with which to fill the interstices between the pebbles.

Thus the theory of composing the best concrete appears simply to involve the same consideration as that of making the strongest mortar with the same materials; that is, with the same kind of lime and sand employed for each purpose.

The pure kinds of lime from Dorking, Reigate, Merstham, Halling, Higham, Frindsbury, and other places in the neighbourhood, although not of such strength as the lime burnt from the lias, the carboniferous, and other hard limestones, will take more sand than the latter in combination with it in the state of mortar.

For instance, the chalk limes will usually take three times their bulk of moist sand, while the lias lime will not make good mortar if more than twice its bulk of sand be used; the quantity for the carboniferous limestones being about a mean between the two, or two and a half times its own bulk measured in a moist state, the lime being measured before slaking. This difference in the proportion of sand to make mortar with different limes appears to depend on the relative purity of the limes themselves.

Thus, the chalk, being the purest carbonate of lime with which we are acquainted, requires a greater quantity of sand than almost any other kind of lime; while

the lias, being an argillaceous limestone, would be overloaded and adulterated by a greater mixture of sand than here stated. It has been already noticed that when the sand used for mortar is of a coarse quality, a less quantity must be mixed with the lime than if the quality be finer; and this axiom will be found in perfect accordance with a law which would regulate, on very sound principles, the quantity of sand which should be used in making mortar, namely, that the sand should measure in a moist state as much, and no more, than when thoroughly incorporated with the quicklime and water. Should the sand for making mortar be too coarse, it will evidently not enter into that intimate combination with the lime which is necessary to produce the requisite adhesiveness and solidifying property of mortar. The cement or mixture containing too great a proportion of large sand may be considered as a putty uniting the coarser particles together, and if the quantity of fine sharp sand in combination with the lime is insufficient, the strength of the mass will not be greater than that of putty, or dried clay, as already stated. It would seem, however, that if fine sharp sand be used for the mortar in sufficient quantity, the addition of a certain proportion of larger particles can do no harm, except when the mortar is required for fine joints of brick-work or exterior masonry; in fact, when much rubble backing occurs in any building, it would be an economical plan to mix small pebbles with the mortar: by this plan a saving of lime could be effected, and the building would in every respect be as strong as if pure mortar alone had been

used. This exception, of course, does not in any way apply to concrete, where the mortar, consisting of the sand and quicklime incorporated with water, is used to cement the pebbles together. It is probable that grains of sand larger than a pin's head do not combine chemically with the lime, so that in all concrete a certain portion of the sand ought to be quite fine. Too much importance cannot be attached to the necessity of using sharp sand, free from earthy particles. The best possible kind of sand which can be used is composed of small grains of angular silex or quartz. The consequence of using sand adulterated with loamy and clayey particles is that the mortar will appear fat, as it is technically called by the workmen, while its real quality will be something intermediate between that of mud and good mortar. Several of the reaches in the river Thames furnish a sand which, although the grains composing it be smooth and rounded, is yet an excellent sand, and much esteemed by builders on account of its purely silicious nature, and its freedom from admixture with particles of clay, mould, or any other foreign ingredient. The London builder has also the advantage of being able to procure from the Thames sand of any degree of fineness or coarseness, according to the place from which the sand is taken; the principal circumstances which regulate the quality either of sand or gravel being the velocity of the current where it is found, and the nature of the streams falling into the river in the neighbourhood of the various beds of sand and gravel. To speak practically of the mixture of sand in making concrete, we

should say that none of the particles ought to exceed the size of a barley-corn, and that a greater portion of the sand should not exceed half that size, while another portion of the sand, equal in quantity to the other two portions united, should consist of grains of sand not so large as a pin's head;—then of such sand a quantity varying from two and a half to three and a half times the bulk of the quicklime may be used in making concrete. The lime should be measured in powder before slaking, and the sand should be measured in a moist state. With respect to the quantity of gravel or pebbles that should be used in the concrete, taking for our guide the law already laid down, that the lime and sand should exactly fill the interstices between the stones when laid as close as they can be without any admixture, it will be seen that such a quantity of lime and sand could be ascertained by experiment in the following manner: namely, take any cube measure of the pebbles from which the sand has been entirely separated, then, by pouring as much water into them as will entirely fill the interstices, the cubic measure of this quantity of water will give the proportion of lime and sand which should be used to any given bulk of the pebbles. It will be found, as the result of this experiment, that the quantity of water required to fill the interstices of the pebbles is something between one-third and one-half the cubical contents of the pebbles, according to the size and shape of the latter. Supposing the quantity to amount to one-half, and the sand, as previously determined, is to be equal to three times the quantity of lime, (which does

not increase the bulk of the concrete when properly incorporated with the sand,) the proportions of these ingredients,—lime, sand, and gravel,—will be as one, three, and six; that is, the concrete should consist of one part by measure of lime, three parts of sand, and six of gravel. Next, suppose the interstices of the gravel or pebbles to be equal to one-third their bulk, then the quantities of each ought to be—lime one, sand three, and gravel nine, parts. Many concretes have been prepared according to each of these sets of proportions, but it will probably be found that the mean of the two, or the proportions of one, three, and seven, or seven and a half, will form a very excellent concrete with pebbles of the average size and lime of the average quality used about London. If, however, lime be used which only takes two portions of sand to one of lime to convert it into good mortar, the proportions for concrete will then be (where the vacuities are equal to one-half the cubical contents of the pebbles)—gravel four, sand two, and lime one; and when gravel is used in which the interstices have been found to measure one-third of the whole bulk, then the proportions will be six of gravel, two of sand, and one of lime.

Various methods of slaking lime are practised, and some engineers, particularly in Scotland, invariably require that the mortar shall be kept for some time after being mixed, before it is allowed to be used in the work. Even in concrete under heavy buildings it is desirable that the lime should be well slaked before being put into the foundation, because the bursting and expansion which

attends the slaking of the refractory portions of lime disturbs the whole mass of concrete, and prevents it from setting into a solid substance, as it ought to do. The practice of grinding the burnt lime into a fine powder before slaking is so very judicious that it should always be adopted: when thus ground, the lime should be equally covered over with the mixture of sand and gravel, in order to prevent, as much as possible, the escape of steam during the process of slaking; water may then be regularly poured on the heap from a watering-pot, and after allowing it to remain in this state about five or six hours, which will be sufficient time for slaking the ground lime, the whole must be carefully mixed together and thrown into the foundation as quickly as it can be got ready.

General specifications for concrete may run thus :—The gravel may be taken out of parts of the river Thames, or from other places hereafter described, but to be of such quality as the engineer shall approve of; the lime to be made from the lias formation, from the carboniferous limestone, from the under beds of the lime quarries of Merstham or Dorking, or from such other hydraulic limestone as the engineer may be satisfied with. If the Thames gravel cannot be conveniently procured, pit gravel, if properly washed, broken, and sorted, may be admitted, also broken flints, stones, and pebbles of any description, lime-siftings, broken bricks, or hard chalk, in the proportion of one-third of the latter to the whole bulk used. None of these materials to be larger than will pass through a ring of three inches diameter, and

they must be entirely cleansed from all clay, soil, or earthy matter of every description, but must be mixed with about one-third their whole bulk of pure sand, more or less, as the engineer may determine. The sand to be sharp and clean, and of the size and quality before described. The lime to be taken fresh from the kiln, and, after being ground in a mill, to be covered with the sand and gravel, and altogether prepared and used as already detailed.

Great difference of opinion prevails amongst architects and engineers as to the depth of concrete which should be adopted in the various situations where it has to be used as a foundation. To determine this it is of course necessary to be well acquainted with the nature of the strata on which the concrete is to rest, and even after knowing this it would be exceedingly injudicious to determine on any thickness which should be hazardous and doubtful. It may be safely stated, that the greatest depth will always be necessary where the foundation is of unequal solidity. Even the softest and most yielding substratum, such for instance as a bed of elastic peat, provided its compressibility be uniform over the whole base, will require a less thickness of concrete to support a building resting upon it than if the same building stood on concrete partly resting on peat and partly on a more solid stratum. As a general guide, the depth of concrete ought in no case of bridge or viaduct building to be less than three feet thick, and it will rarely be necessary to use a greater depth than six feet. Concrete being, however, much cheaper than any other kind of artificial

foundation,—whether of stone, brick, or wood,—there can be no impropriety in putting any thickness that is necessary to raise the building from a solid base to nearly the surface of the ground, from which level a better description of building becomes necessary.

In the course of the preceding papers, while considering separately the various kinds of foundations, we have been led to remark on a few of the more common causes of settlement and failure in heavy buildings. Thus we noticed that timber platforms, placed under a building, were subject to decay from the alternations of drought and moisture, from which arises the necessity of well observing whether the timber will be exposed permanently either to one or other of these states, because the changes from one to the other are exceedingly injurious to every kind of wood-work.

The agency of water in producing dangerous consequences where a building is founded on sand has also been pointed out, and on this subject it will only be necessary to add our conviction of the great caution which should be used by engineers in founding either upon an existing quicksand, or upon any sandy stratum which may by the future operations of water be converted into this state. We have further noticed, as a point of great importance in all foundations, but particularly those of stone, where unequal settlement is more injurious than in concrete foundations, that it is most essential to guard against the unequal solidity of the ground to be built upon, because it is in cases of this

kind that unequal settlements of the building take place, and occasion fractures alike injurious to the appearance and to the stability of the structure.

The settlement in the wing wall of Gloucester Bridge affords an instructive example of the consequences which have followed an unfortunate inattention to this subject. One of the wing walls on the Gloucester side of this bridge is fractured from its base to the top of the parapet, where the opening is nearly three inches wide.

The cause of this settlement, as explained by Mr. Telford himself, appears to be that the abutment was founded thirty-three feet below the ground surface on a bed of strong coarse indurated gravel, while the wing walls joining the abutment, although they had to support the weight of a heavy embanked approach, were founded only ten feet below the surface on a bed of soft blue silt. In addition to this, the abutment had a timber platform placed upon a pavement of rubble stones in the bottom before the building was commenced, whereas the platform was entirely dispensed with in the foundation of the wing walls. The consequence of this imperfection is that the alluvial soil has given way, by reason of which the wing on one side, if not on both sides, at the Gloucester end of the bridge, has receded from the abutment, and a very large and unsightly fissure has taken place vertically in the line of junction. Whatever could have led to the unfortunate error which was here committed, of admitting so great an inequality in firmness of base for the wing wall as compared with that for the abutment, we are quite unable to conjecture, and the

circumstance is the more remarkable when we consider the known ability and experience of the contractor who was entrusted with the work. Mr. Telford, with a generosity worthy of his character, took the blame upon himself of being influenced by what he justly terms the injudicious parsimony of omitting a platform and piling for the wing walls. It is probable, however, that Mr. Telford himself, being at a distance from the place, was not so fully acquainted with the circumstances as he might otherwise have been, and as he afterwards necessarily became. Upon the whole, the case is worth recording, in order that it may operate, in Mr. Telford's own words, as "an useful caution to practical engineers."

We shall now relate an instance where an extensive building, consisting of several arches, was destroyed in a way which at first sight appears scarcely to be accounted for on ordinary grounds, but which, we think, may fairly be attributed to the action of water. The case in question is of recent occurrence, and as the mention of the particular work might give rise to hostile feelings on the part of the engineers and others concerned, we refrain from making public the locality. It will be sufficient for the purpose we have here in view to explain that the building was a viaduct, of considerable magnitude, placed across a valley, in the centre of which flowed a small stream. The upper stratum, on which the structure was raised, consisted of loose earth and rubbish taken from the foundations of old buildings. The depth of this made ground was about five feet; under this was a bed of clay averaging about six feet in

thickness, and under the clay a stratum of sand fourteen feet thick. The design for the building was well chosen, the proportions of all its parts were unexceptionable, and the workmanship of the brick-work and stone, laid partly in mortar and partly in cement, was of the very best description. The work had gone on progressively, and in a way quite satisfactory to all engaged, until nearly the last arch was thrown, when all at once two of the other arches fell, and a third was seen to be in so much danger that it had to be taken down in order to prevent a similar fate. All the piers and abutments had been founded on the bed of clay mentioned above, and we are strongly of opinion that the sinking of the piers and consequent falling in of the arches was occasioned by the action of the stream upon the bed of sand beneath the clay. The stream, it is true, flowed over the clay, which formed its bed, and was not, in the immediate neighbourhood of the viaduct, in contact with the sand; but we think it very clear, from the result which followed, that at some place not very far distant the water had a communication with the sand, by means of which the latter was set in motion, so as to leave an unsound space below the stratum of clay. A very strong proof that the mischief must be attributed to some change wrought below the clay may be derived from the fact that the clay on which the defective piers rested was only four feet six inches thick, while the general depth under the rest of the building was six feet. Besides this, under one of the piers which gave way the clay was dug out to the depth of one foot nine inches, and a bed of concrete

of the same thickness substituted in lieu of the clay, and under another of the piers, which also sunk, a layer of concrete one foot in thickness was laid. Thus the whole thickness of clay between the bed of sand and the bottom of the one pier was two feet nine inches, and between the sand and the bottom of the other was three feet six inches. These two piers gave way, while the other piers, founded on the undisturbed bed of clay six feet in thickness, remained firm and solid. It seems, therefore, that the sand, which extended under the whole building, being saturated by the water, which probably obtained access during a time of flood, and which found an outlet to escape when the flood subsided, was not able to support the pressure of the clay when weighted by the building above, and it followed that the piers which were founded at the least depth above the sand sunk down with the clay, and caused the arches to fall as already described. On the other hand, the piers placed upon the full thickness of six feet of clay stood without sinking; so that here we may suppose the mass of clay, although partially hollowed by the defection of the sand, was yet sufficiently solid to support the weight of the building without material settlement.

Almost too obvious to need comment is the error which, as after-experience established, had been committed in the foundations of this viaduct. It will occur to the most unpractised student that had the clay been entirely taken out in the places where its depth was judged to be insufficient, instead of attempting to strengthen it by concrete, and had the concrete been laid directly upon

the sand and carried up to near the surface, where the building might have been commenced, we should not have had the present case of failure to cite as an example for future works. It might be worth while, however, in determining on the foundations of a viaduct consisting of a number of arches inconsiderable in span, to consider the cost of inverted arches between the piers as compared with that of concrete placed under them. The inverted arches should be eighteen inches thick, extending from pier to pier over the whole base of the structure, and the broad substantial bearing they afford will either entirely prevent settlement or will render it so uniform that no danger need be apprehended.

In confining our attention now to the general subject of settlements, independent of the particular nature of the foundation, we must not omit to notice the importance of firm solid backing for the walls of buildings which are required to withstand the pressure of earth. The wing walls and abutment walls of bridges form an important class of revetment or retaining walls of this kind, and, next to the actual preparation of the foundation on which they rest, the backing behind the walls claims the care and attention of the engineer. The object intended to be effected in the backing of walls is that of rendering the earth pressing upon the wall uniformly solid, in order that no irregular pressure may be allowed to act upon it. The most common form of imperfect and unsound backing is that in which water is allowed to lodge in hollows and crevices behind the wall, and thus to exert a pressure greatly in excess of that which the

wall is calculated to withstand, and greater than would ever occur in the case of dry earth behind the wall. From this we are led to observe the necessity of rendering the backing in a great degree water-tight, in order that it may act as a puddle-dam to prevent water from reaching the back of the wall. In considering the best material to be used for this kind of puddle backing, the same observations made in a former paper upon clay and other substances with reference to the puddle for coffer-dams will equally apply in the present instance. The cracks and fissures which attend the drying of clay, when much exposed to the sun and wind, are so exceedingly dangerous, as affording lodgement for water to press against the wall, that there is every reason to expect, at some time or other, fractures and dangerous settlements in walls which have been backed entirely with clay. It frequently happens that the wing walls of bridges which have stood for a long time, perhaps for many years, without showing any sign of fracture or decay, suddenly give way to pressure from behind, and in such cases the misfortune is usually attributable to the use of clay backing. Either the clay will have cracked so as to admit water throughout the crevices, or it will be found to have contracted in bulk and to have shrunk away from the wall, in each case leaving it exposed to the pressure of a column of water equal or nearly equal in height to that of the wall itself.

Although it is, therefore, very injudicious to adopt a puddle of pure clay, yet, if no other material can be procured, it will be found an excellent practice to dis-

continue the puddle at about eighteen inches below the top of the wall, and to complete the backing with a course of sand or fine gravel, which, resting on the clay, will serve to keep it moist and prevent it from cracking. It is also necessary in all heavy walls intended to withstand the pressure of earth, that small orifices, about four inches square, should be left at regular intervals in the face of the wall; these should pass entirely through the thickness of the wall, and be placed six or seven feet apart longitudinally, and about three feet apart vertically.

A grout puddle, which is very commonly used for the backing of lock and dock walls, is formed by a mixture of gravel, chalk, and a slight portion of clay or other adhesive material, with just sufficient water thoroughly to incorporate the whole mass. Puddles of this kind are far more to be depended on than clay puddles, since they are generally more water-tight, and are never known to crack.

In fixing the foundation of any building on a clay bottom, great care should be taken to place it sufficiently deep to be out of the reach of frost. This should particularly be attended to in the wing walls of canal and railway bridges, which in cuttings are usually made to step up the slopes. If the frost be allowed to act on the foundations of these bridges, an expansion and consequent fracture will take place, frequently to the extent of shattering and destroying the whole wall.

We had intended, in the present paper, to introduce some notices of the most important building stones and quarries from which stone is procured for the purposes

of bridge building in this country, but this has been rendered entirely unnecessary by the appearance of a very valuable Report from the Commissioners appointed by the Treasury to investigate the best kind of stone to be used for the New Houses of Parliament.

This report is in every way worthy of the talents and scientific attainments of the gentlemen who have prepared it, and probably a more useful and valuable body of information than that contained in the report itself, and the tables which accompany it, was never before placed within the reach of the engineer. Some idea of the labour and pains bestowed by the commissioners in the execution of their arduous yet interesting and instructive task may be derived from the fact that they have visited the localities, and made themselves perfectly acquainted with the nature and properties, of more than one hundred different kinds of building stone in England and Scotland. The information collected during their researches has been embodied in several tables, one of which describes each quarry which has been examined, and furnishes the following particulars:—The mineral designation of the stone; its component parts and colour—weight of a cubic foot of stone in its ordinary state—the entire depth of workable stone in the quarry; description of the beds and size of the blocks that can be procured.—Where known or reported to have been employed, and general remarks—prices of block-stone at the quarry; description and cost of carriage to the pool of London; cost of stone delivered in London per cubic foot; and cost of plain rubbed work as compared

with that upon Portland stone in London, per foot superficial, Portland being taken at 1·0.

Another table describes a vast number of buildings which the commissioners have also visited, in order to inform themselves as to the durability of the stone composing them.

The two last tables contain the results of experiments on the cohesive powers of a great variety of specimens, with chemical analyses of the most important by Professors Daniell and Wheatstone.

These valuable tables, with the report, are published in the Civil Engineer and Architect's Journal for September, October, and November of the present year, a very able and useful publication, to the editor of which the profession of engineers and architects are indebted for a vast accumulation of practical and scientific information.

In concluding the last of this series of practical papers, we propose to take a brief review of the construction of contracts for building bridges, in so far as they relate to the engineering of these works Any observations we shall make on this subject will apply equally to the contracts for constructing most other engineering works. In fact, this general application is necessary, as far as the bridges on railroads are concerned, because, in railway specifications, the bridges are included in the same contract with many other works; and, of course, the general clauses in the contract deed apply to the bridges in common with the other works.

The object of these general clauses is, in the first

place, to bind the contractor to execute certain works according to the designs and specification of the engineer. The designs are affixed to the contract in the shape of plans, sections, and general drawings, explanatory of the works to be constructed, and the specification follows the contract as a schedule, which is occasionally referred to as the technical instrument for guiding the operations of the contractor.

In the next place, the general clauses provide for any alteration in the original design of the engineer, and give to the latter an arbitrary power of directing the contractor to make any alterations which he (the engineer) may think proper. The way in which alterations, whether of increase or diminution in the amount of works, are to be estimated is usually regulated by a schedule of prices which the contractor is required to furnish as a part of his tender to execute the work, and this schedule is also appended to the contract deed.

Another series of clauses, usually introduced into railway contracts, confers upon the engineer an absolute power over the entire operations of the contractor, so that not only the method of construction, but also the times and seasons for working, the amount of labour to be employed, the tools and machinery to be used, are all subject to the approval of the engineer.

In order that all these clauses may be enforced for the benefit of the company, (who, as parties to these contracts, most assuredly take care to render their own the stronger side,) they are empowered by another clause, at any time they shall think proper, to set aside the con-

tract, to stop the contractor in his work, and take possession of all his tools, materials, and property of every kind which may be on the site of the works. By this act of seizure the company take the work into their own hands, and they may proceed to complete the same entirely independent of the contractor, from whom and from his sureties the company, after having executed the works under their own management, may recover any excess of cost which shall have been incurred over and above the original contract sum.

We shall now proceed, in detail, through the principal clauses of a contract deed for building a bridge.

Passing over the preamble, which is frequently a mere legal form and without interest to the engineer, the first clause sets forth, that in consideration of the sum or sums of money thereinafter agreed to be paid, the contractor, for himself, his heirs, executors, and administrators, doth covenant, promise, and agree that he will begin, and in a substantial, perfect, and workmanlike manner, and to the satisfaction, and according to the directions, of the engineer, build, erect, complete, and finish the bridge in the manner thereinafter mentioned, together with the approaches thereto, and all and every other work and works, according to the plans agreed upon by the parties to the contract, and under and subject to the directions, rules, regulations, explanations, and restrictions mentioned or referred to in the specification contained in the schedule thereunder written or thereunto annexed, with such alterations (if any) as from time to time shall be directed by the engineer.

The succeeding clauses go on to provide that the contractor is to furnish, at his own cost and charges, all the materials incident to or necessary for executing the works.

Then follows a clause which places upon the contractor the responsibility of the whole design, in case of failure, from any cause whatever. The clause in bridge contracts is frequently to this effect,—"That the contractor shall bear all loss, risk, and responsibility whatsoever attending the execution of the works hereby contracted to be performed, and shall and will, at his own expense, forthwith, and without any delay, make good all damages of every description, by floods or otherwise, which may happen to the said works, or any part thereof, during the progress of the same."

We would humbly suggest, that alike for the credit of the engineer, and in justice to the contractor, this clause ought to be modified wherever the former feels that confidence, which we presume he always ought to feel, in his own designs. When certain works are specified, and their mode of execution particularly detailed, and where the contractor has faithfully followed the instructions, and adhered to the specification of the engineer, he ought at the most to be answerable only for those accidents and casualties which arise from sudden visitations of the elements, such as damage by lightning, earthquakes, or floods.

The contractor's tender may be made sufficient in amount to cover all probable risk from accidents of this kind; but it is a more difficult affair to estimate the

chance of failure from a new or partially untried design; and if the engineer, who is himself the author of the design, should shrink from the responsibility of constructing it, with how much greater reason may the contractor plead the hardship of a position which renders him liable to such serious risk!

The next clause specifies the time by which the work is to be completed, and occasionally imposes a penalty, to be paid by the contractor, for each week that the actual time of execution shall exceed the time specified. The contractor is also very frequently bound to keep the bridge in repair for one, two, or three years after completion, in order that, in case of any failure from imperfect construction, the expense of restoration may be borne by him, as provided in a former clause. The contract sum is then set forth, and the times and mode of payment are either mentioned in this place or at the end of the contract.

The next clause relates to alterations in the original design, and is more or less severe upon the contractor in different contracts. One of the strictest forms we have seen, as giving to the engineer entire and absolute power with respect to alterations and payment to the contractor, provides, "That, in case at any time or times, during the progress of the works, the engineer shall think proper to cause any alteration in, or variation from, the original plans and specifications to be made, either by increasing the said works, or the scale or magnitude thereof, or omitting some part thereof, or diminishing the said works, or the scale or magnitude

thereof, or altering the quality of any part of the said works, or the materials to be used therein, or otherwise howsoever, the contractor shall and will execute, perform, and complete the said works according to every or any such alteration or variation in the manner, and within the time in which the said works ought to be completed, according to the true intent and meaning of these presents, and no such alteration or variation shall vacate or lessen the validity of any of the covenants or agreements herein contained, but such sum of money shall be added to or deducted from the contract sum as the engineer shall estimate to be the value of such alteration or variation."

The contractor by this clause is evidently placed at the mercy of the engineer, with whom it is entirely optional whether any and what allowance shall be made for what are usually termed extra works, and for every other kind of alteration which the engineer may choose to make. The form of the above clause is usually varied in railway contracts so as to make the schedule of prices the standard by which alterations or variations are to be valued; but in many of these contracts it rests with the engineer to determine whether any allowance at all is to be made to the contractor, or whether in lieu thereof a deduction is to be made from the contract sum. Under either form of clause the power of the engineer is absolute; and we apprehend, whatever may be the integrity and general moral excellence of the man possessing such a power, it is too great for him to exercise without the danger of partiality. We have no intention

of dwelling at length upon this particular subject; but let any unprejudiced mind consider on the one hand the engineer pledged to execute his work for an insufficient estimate, with the interest of his own employers to consult, his own credit at stake in many different ways, and on the other hand no possible, or at least no ostensible, object for favouring the contractor, and judge to what side his opinion will incline. Let the contractor in such a case value his chance at what he pleases, he will over estimate it; the final issue will probably be that he must content himself with having bought experience at a price which he can probably afford to pay only once in his life.

In the specification which follows the account of Hutcheson Bridge in another part of this work will be found the fairer and more reasonable stipulation with regard to extra works, which entitles the contractor to be paid for these upon the certificate of the engineer, according to the schedule of prices.

The next clause empowers the engineer to reject any materials brought upon the site of the works, and the contractor is bound to remove such rejected materials, or pay for the expense of removal by workmen to be employed by the engineer.

In a succeeding clause the engineer is further empowered, in case he should disapprove of the workmanship, materials, or execution of any part of the works, to order the contractor immediately to take down and re-execute or alter the disapproved part or parts to his (the engineer's) satisfaction. And should the contractor refuse

or neglect to comply with this order within the space of one week, the engineer may employ workmen to take down, alter, amend, or rectify such disapproved parts of the work. The cost of employing these other workmen, and defraying their bills for labour and materials, to be deducted out of the balance then due to the contractor; and should this balance be insufficient, the contractor or his sureties must make good the deficiency.

The clause by virtue of which the company may invalidate the contract and take the work into their own hands is variously modified in different deeds. As this clause is very important, we shall give the form of it in full.

"In case the contractor shall refuse or neglect to perform the (aforesaid) works, or any of them, in manner hereinbefore in the said specification mentioned, or to obey and comply with any orders or directions to be given by the engineer, or in case at any time during the progress of the works there shall appear to the engineer to be any unnecessary delay in the carrying on of the works, or any part thereof, either by not employing a sufficient number of workmen, or otherwise howsoever, or in case any of the works shall not be performed to the satisfaction of the engineer, or shall not be finished within the time hereinbefore mentioned for completing the same, then, and in such case, it shall be lawful for the company to revoke and make void this present contract, and every clause, matter, or thing herein contained, so far as the same relates to the part of the said works which shall not have been performed, and the

contractor shall be entitled only to such part of the said sum or sums of money hereinafter agreed to be paid as shall be estimated by the engineer to be the value of the part of the works which shall have been performed."

This clause, in the case of railway contracts, is rendered more severe upon the contractor, inasmuch as " the company, if they shall think fit so to do, may seize and take possession of the works, and of all or any part of the materials, engines, machinery, implements, and utensils provided by the contractor for the execution thereof, and may employ any other person or persons, either by contract, or measure and value, or otherwise, or by themselves, their engineers, servants, agents, workmen, and others, may proceed with and complete the said works according to the terms of the specifications, and the true intent and meaning of these presents, &c."

The contractor is frequently prohibited from making sub-contracts, or from under or sub-letting any part of the works without the consent of the engineer; but of course this particular prohibition is quite unnecessary wherever the engineer is armed with the arbitrary powers conferred by preceding clauses, and where the company may invalidate the contract whenever the work is not proceeding to their satisfaction.

As it is far from unlikely that disputes may arise relative to the meaning or interpretation of the drawings and specification, the engineer is appointed in case of any misunderstanding to construe, interpret, and explain; and again, in case of discrepancy or disagreement between the drawings and specification, the engineer's de-

cision as to which of these shall be followed is made binding on the contractor.

In railway contracts there are usually some clauses of minor importance, such as that which obliges the contractor to make accurate returns to the engineer of the number of workmen of all kinds employed on the work from week to week. In addition, also, there is one which vests in the company, during the progress of the work, all the property of the contractor which shall be brought upon the site of the works for the purpose of being used therein, and the contractor is prohibited from removing any part of such property except under the sanction of the engineer or that of the company.

The contract concludes with the covenant, on the part of the company, that they will pay to the contractor the sum for which the works are to be performed at such times, and in such proportions, as are therein set forth.

Such is the usual form of contracts for building bridges in this country; and subject to stipulations differing slightly, or not at all, from those recited above, many of our largest works have been executed. The construction of the deed being commonly managed by gentlemen of the law, it is usual to find the subject matter very much amplified, and of course a great many unnecessary words and phrases are employed; but the preceding extracts and outlines of clauses may be taken as the real substance of the most complete contract deeds for the building of bridges.

T. H.

# SOME ACCOUNT

OF

# HUTCHESON BRIDGE OVER THE CLYDE AT GLASGOW.

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Most of our readers are aware that among the rivers of Scotland the Clyde ranks next in importance to the Tay, which discharges more water than any other British river, and that Glasgow, to which the Clyde gives such great commercial importance, stands highest in the scale of population among the cities of Scotland.

Prior to the year 1772, when the Broomielaw Bridge was opened, the thoroughfare across the river at Glasgow was confined to the narrow bridge opposite Stockwell Street, built by William Rae, bishop of Glasgow, in 1345, which, after the Reformation, became the property of the corporation of that city.[1] This, like almost all ancient bridges, was incommodiously narrow,

[1] The *name* of the old bridge, and by which in some forgetfulness of the circumstances the trustees have recently designated the new bridge at Jamaica Street, is "The Bridge of Glasgow;" the first Act of Parliament, 32 Geo. II., for building the new bridge, expressly narrating the circumstance of the Stockwell Bridge being so called.

and its roadway steep. It has, however, undergone several alterations by which its breadth has been extended and its approaches greatly improved. From the imperfect state of the navigation between Greenock and Glasgow, and the rising importance of Paisley, the traffic between these places along the old bridge became so very inconvenient that the new bridge at the Broomielaw was found indispensable. But this bridge has in its turn become too narrow for its traffic, and has been removed and rebuilt in a much more spacious and elegant style. Between these two stone bridges a timber bridge was erected in the year 1831, agreeably to plans by Mr. Stevenson, engineer for Hutcheson Bridge, in a style accommodated to the flat nature of the banks, and with a level roadway. Higher up the river we have the recent erection called the "Hutcheson Bridge," situated immediately above the Court Houses, forming a direct *entrée* to the city from Ayrshire.

Hutcheson Bridge is so called from two brothers, George and Thomas Hutcheson, of Lambhill, the elder of whom died in 1640, and the younger in 1641, leaving large sums to endow an Hospital for the relief of the aged and infirm, and also for the education of youth. The magistrates and clergy of Glasgow were appointed trustees, and instructed by the donors to lay out the money "upon the best and cheapest arable lands that can be got near this burgh." In pursuance of this mandate the trustees purchased the half of the barony of Gorbals, which is now, perhaps, among the most important appendages of the city of Glasgow.

The feuars of Hutcheson-town were bound by their titles to pay an additional feu or ground-rent upon the erection of a bridge over the Clyde leading immediately into their lands, and the trustees for the Hospital, along with others, entered into contract with a Mr. Roberton to build the proposed bridge, to consist of five arches, for the sum of £6,000; the extreme length was to be 400 feet, and its width within the parapets 26 feet. The specification of the work, however, was unfortunately defective, and wanted solidity proportioned to the force of the current and depth of foundation suited to the soft nature of the bottom. The work, however, went on, and all the arches were turned, and the retaining walls nearly finished, when, on the 18th November, 1795, in one of those floods to which the Clyde is incident, the fabric was unfortunately swept away. Mr. Stevenson, the engineer of the present bridge, was at this time in Glasgow, and when the alarm was given he instantly repaired to the spot, where the river presented a sad scene of ruin. The arches had burst upwards with so much force that the key-stones were thrown to a considerable height above the arches. The northmost arch was carried away at about two o'clock in the afternoon, and before sunset the whole were gone, and the bed of the river immediately below the site of the bridge, and where the medium summer level, at low water, is only 18 inches, was then scooped out to a great depth, in some places even to 28 feet.

In a work treating of bridges, details of this kind

become interesting as matters of history, and professionally valuable to the engineer. A question immediately arose between the contracting parties as to who were to be the losers in the business. The contractor had lost the labour of at least two years, and the trustees had by this time advanced the sum of £4,000 on account of the works. It was alleged, on the part of the contractor, that the magistrates, as trustees, had refused or delayed the erection of a retaining wall, connected with the abutment of the bridge on the city side, and the stability of the bridge was said to have been endangered by a large quantity of rubbish, improperly deposited at the northern abutment. The question as to the cause of the damage was at length submitted to the late eminent Mr. Rennie, engineer, and the late Mr. Wilson, architect, who reported on the subject, and the contractor was ultimately relieved from his obligations on refunding the sums which had been advanced him, without interest.

After this accident nothing further was done towards the erection of Hutcheson Bridge for about twenty-three years. But the increasing inconvenience to the lands of Hutcheson Hospital, and the public generally, having been brought under the notice of the trustees for the Hospital, the proposal of a new bridge was warmly espoused by Robert Dalgleish, Esq., as Preceptor or President of the Hospital, and the Very Rev. Principal Macfarlan, of the College, one of the committee, who always took a lively interest in the recompletion of the meausre. It was accordingly finally resolved on in

the year 1828, a new Act of Parliament being expressly obtained for this purpose, and also for opening a new road or street in connexion with the bridge, through the Hospital's estate, to the great turnpike road between Glasgow and Ayrshire. Although the detailed specification of this bridge is given in this work, yet, viewing Hutcheson Bridge as one of the most considerable of Scottish bridge-works, a descriptive account of it is here given, for the use of those who may not have occasion to look into these technicalities. The design of this bridge was made by Robert Stevenson, Esq., of Edinburgh, and the works were contracted for and executed by Mr. John Steedman, now local engineer for the county of Donegal. As a piece of masonry, Hutcheson Bridge stands unrivalled; and in an engineering point of view it may be regarded as remarkable from the difficulties which attended the arduous works for establishing the foundations of the piers and abutments of the bridge.

This bridge consists of five arches, which are segments of a circle whose radius is 65 feet. Two of the arches are 65 feet, two 74 feet 6 inches, and the middle arch 79 feet, in the span; with versed sines of 8 feet 8 inches, 11 feet 9 inches, and 13 feet 4 inches, respectively. This rise is proportionally less than perhaps that of any other arches of the same extent and figure in the kingdom. By this means, although the banks of the river are low, approaches are obtained on either side with very little embanking or damage to the adjoining streets, and a line of draught is obtained at the easy rate of one in

thirty. Although, for our part, we are inclined to think the beauty and strength which this bridge possesses would make its own features and proportions better shown without the foreign aid of any ornament at all, yet the elevation, as will be seen from Plate No. 27, is adorned with columns and niches, the columns resting on the piers and supporting an appropriate entablature which runs over the arches throughout the whole length of the bridge. The niches were designed to contain statues of the benevolent founders of the Hospital on the one side, and figures emblematical of the objects of the charity on the other. The columns are so arranged as to be capable of supporting footpaths, should it be found advisable to widen this bridge, as has been the case with those further down the river. The carriage-way is macadamized, and the footpaths, on either side, are laid with Caithness pavement, which, besides being remarkable for cleanliness, is exceedingly durable.

The ceremonial of laying the foundation stone was performed by Robert Dalgleish, Esq., the Lord Provost of Glasgow and Preceptor of the Hospital, on the 9th day of October, 1829, and the works were completed in 1833. The contractor had unfortunately not only a succession of bad seasons to contend with while this work was in progress, which kept the river much in flood, but when the ground was opened for the foundations of the abutments and piers it turned out to be extremely untoward. When the foundation pit of the northern abutment had been excavated to the specified depth, its appearance was so unfavourable that the

trustees, in some alarm, sent for the engineer, to consider whether the excavations for the foundations were really practicable to the specified depth. The appearance of the bottom of the excavated part was no doubt most forbidding, the whole being in a state of motion, with perhaps a hundred springs of water boiling up. The soil in fact was what is called *running* sand, mixed with minute portions of silt. It was therefore found necessary to enlarge the platform for the foundation course of that abutment, to increase the number of bearing piles, and to drive the outer row of them close, like sheeting piles. It was at the same time determined not to draw the inner row of the sheeting piles of the coffer-dam, but to drive them home, as an additional security to the foundation. On the engineer's reporting the propriety of these extra and precautionary works, they were at once acceded to and ordered by the trustees.

Great difficulty was found in keeping under the springs of water which boiled up over all the foundation with great force, and also in the close driving of the extra piles, which often recoiled to the stroke of the piling machine. The foundations of the former bridge were likewise found very troublesome to remove. The piles, as will be seen from the annexed specification of the bridge-works, were 9 inches square and 18 feet long, driven to the depth of 9 feet under the level of the summer water mark; at which level the platforms were laid. The ground was found to be of much the same nature all the way across the river, though it rather improved toward the southern side. But here

also work of considerable expense was required, for the safety of the lofty houses of the contiguous street. It became necessary to drive a row of piles landward of the southern abutment of the bridge, as a circumvallation, as shown in Plate 29.

The best materials which the sandstone quarries in the neighbourhood of Glasgow could produce were selected by Mr. Laidlaw, the resident inspector, which were afterwards carefully dressed and built under his superintendence. The mortar was prepared in like manner from the water-limestone quarry of Arden in Renfrewshire, and from Aberthaw in Wales.

Hutcheson Bridge, on the whole, is believed to contain the best collection and the largest sized materials of any of our public works of similar extent. This bridge, being first in order, has the burst of the floods and ice to bear and regulate for the City Bridge below; and from the circumstances attending the former bridge, with the engineer's knowledge of the nature of the river, he did well to provide for a massiveness and strength of specification, as indispensable to the permanency of structures similarly situated. The engineer of the splendid bridge at Bourdeaux, which Mr. Stevenson had carefully considered while it was in progress, and which presents various points of similarity, has constructed a great extent of additional masonry, forming two archways within the walls of the bridge itself, chiefly for attaining the same object.

Independently of the expense of the Act of Parliament, engineering charges, and other contingencies, the contract

price and extra works of Hutcheson Bridge amounted to upwards of £23,000. And though the contractor did not find this a work of profit, yet with his perseverance and unflinching steadiness, in spite of many harassing difficulties, he succeeded in completing one of the best pieces of bridge masonry that has been executed in the kingdom.

# SPECIFICATION

OF

# HUTCHESON BRIDGE OVER THE RIVER CLYDE AT GLASGOW.

BY ROBERT STEVENSON, OF EDINBURGH,

CIVIL ENGINEER.

*Edinburgh, April,* 1828.

HUTCHESON Bridge is to be built across the River Clyde at Glasgow, in a line with Crown Street and the front of the Court House, agreeably to a design made by Robert Stevenson, Civil Engineer, Edinburgh, who, previously to the commencement of the work, will more precisely determine and point out the site and line of direction of the bridge, which is to be executed and completed as hereinafter described in this specification, and represented in the accompanying drawings, signed by him as relative hereto.

## EXTENT.

The bridge is to consist of 5 arches, having an aggregate water-way of 358 feet; and including the 4 piers, the whole extent from the front of the abutment at the

impost course on the southern side of the river to the front of the abutment on the northern side is to measure 404 feet, independently of the abutments and landward wing walls on either side. The breadth of the bridge over the soffit of the arches is to be 38 feet. The elevation and works on the one side of the bridge are to be equal and similar, in all respects, to the elevation and works on the other side.

### SUMMER WATER LEVEL.

The surface of the River Clyde being subject to considerable variation of level, from the joint effects of tidal water and land floods, its summer water level, when unaffected by the tide, is hereby fixed at $18\frac{1}{2}$ feet below the arrow point cut upon the Court House near the south-eastern angle of that building, to denote the height of the water of the Clyde during the great speat in the year 1782, which mark is to be referred to in regulating the specified depth of the foundations and other dimensions of the bridge.

### COFFER-DAMS.

It having been ascertained, by boring and mining, that the subsoils of the bed of the river consist of gravel, sand, and mud, to the depth of 27 feet and upwards, it becomes necessary to prepare foundations of pile-work for the bridge; and therefore, to ensure the proper and safe execution of the works, coffer-dams are to be constructed round each of the foundation pits of the two abutments and four piers, of such dimensions as to

afford ample space for driving piles, fixing wale-pieces, laying platforms, pumping water, and setting the masonry; and likewise for the construction of an inner or double coffer-dam, should this ultimately be found necessary.

The frame-work of the coffer-dams is to consist of not less than two rows of standard or guage and sheeting piles, kept at not less than 3 feet apart for the thickness of a puddle wall or dyke, which space is to be dredged to a depth of not less than 9 feet under the level of the summer water mark above described, before the sheeting piles are driven. The guage or standard piles are to measure not less than 24 feet in length and 10 inches square; they are to be placed 3 yards apart, and driven perpendicularly into the bed of the river to the depth of 16 feet under the level of summer water mark, thereby leaving 8 feet of their length above that mark.

Runners or wale-pieces of timber, 9 inches square, are then to be fitted on both sides of each row of the guage piles, to which they are to be fixed with two screw bolts, of not less than one inch in diameter, passing through each of the guage piles. One set of these inside and outside wale-pieces is to be placed at or below the level of summer water mark, and the other set within one foot of the top of each row of said piles; the whole to be fixed with screw bolts in the manner above described. The wale-pieces are to be $4\frac{1}{2}$ inches apart, in order to receive and guide the sheeting piles. This is to be effected by notching the wale-pieces into the guage piles.

The sheeting piles are to be 21 feet in length, $4\frac{1}{2}$ inches in thickness, and not exceeding 9 inches in breadth. They are to be closely driven, edge to edge, along the space left between the walings, and each compartment of the sheeting between the guage piles is to be tightened with a key-pile.

The coffer-dam frames are to be properly connected with stretchers and braces before commencing the interior excavation. Each coffer-dam is to be provided with a draw-sluice, 14 inches square in the void, with a corresponding conduit passing through the puddle dyke at the level of summer water mark. To render the coffer-dams water-tight, the whole excavated space between the two rows of piling is to be carefully cleared of gravel, sand, or other matters, to the specified depth, and clay, well punned or puddled, is then to be filled in and carried up to the level of the top of the sheeting piles. But if it shall, notwithstanding, be found that the single tiers of coffer-dam do not keep the foundation pits sufficiently free of water for building operations, the water must either be pumped out and kept perfectly under by steam or other power, or else excluded by the construction of a second tier of coffer-dam, similar in construction to the first.

For the foundation pits of the two abutment piers on either side of the river it is not expected that more will be required on the landward side for keeping up the stuff than a single row of guage and sheeting piles; but if the engineer shall find other works necessary, upon opening the ground, they must be executed by the con-

inches in diameter, laid in strata not exceeding 6 inches in thickness, and beaten all over with a pavier's beater, weighing not less than 60 pounds; each layer to be well grouted with mortar prepared agreeably to directions under the article "Mortar," and the whole, when finished, to be on a level with the pile heads.

### SILL-PIECES.

Sill-pieces of beech timber, measuring not less than 12 inches by 7 inches, are then to be laid longitudinally upon the site of the foundations and tops of the piles, to which they are to be fixed down with oaken trenails $1\frac{1}{2}$ inch in diameter, and 21 inches in length. Timber of the same scantling and quality is in like manner to be fastened to that part of the piling which is under the cutwaters of the piers, and along the ends and sides of the abutments and wing walls, and is to be half checked at the ends of the sills, and secured to them by oaken trenails.

### UPFILLING BETWEEN THE SILL-PIECES.

The spaces between these timbers or sill-pieces are to be built up flush with their upper side by *one* course of square dressed stones of a size sufficient, both in breadth and depth, to fill up the spaces. These stones are to have a chisel-draught round the edges; they are to be pick-dressed, and set in mortar.

### PLATFORMS.

When the foundations of the piers and abutments,

and so much of the wing walls as may be found necessary, have thus been brought to an uniform level surface, a platform of beech timber $3\frac{1}{2}$ inches thick is then to be laid down transversely, each plank of which is to be equal in length to the full breadth of the foundation at the place where it is laid, and is not to exceed one foot in breadth. The planks are to be set flush in mortar, and fixed to each sill-piece by two iron spikes, measuring one half inch square at the neck and 8 inches in length.

### ABUTMENTS AND PIERS.

The first course of masonry of each abutment is to be laid on these platforms at the depth of 7 feet under the level of summer water mark, and is to measure over all 49 feet in length and 17 feet in breadth between the face next the river and the soffit of the crown of the horizontal arch. The first courses of the masonry of the piers is, in like manner, to be laid at the depth of 9 feet under the level of summer water mark, and is to measure 53 feet 6 inches in length between the extreme points of the cutwaters, and 17 feet 6 inches in breadth. The foundation of each of the abutments is to consist of five courses of squared masonry, of the average thickness of 16 inches, and having an offset of 6 inches all round. The foundations of the wing walls are to have but two offsets on the inward face, and are to be finished in all respects agreeably to the drawings. The foundations of the piers are in like manner to have offsets of 6 inches on each of the first six courses, which are to average 16 inches in thickness.

The height of each of the piers, from the highest offset to the top of the impost course, is to be 10 feet 6 inches, with a batter or slope of 3 inches all round. The breadth of each of the middle piers immediately above the offsets is to be 12 feet 6 inches, and at the top, exclusive of the projection of the imposts, it is to be 12 feet, while the breadths taken at the same place on each of the landward piers are to be 11 feet 6 inches and 11 feet respectively. The shafts of the two middle piers are each to measure 47 feet 8 inches between the extreme points on the level of the highest offset, and on the level of the impost course 47 feet 6 inches, while the dimension at the same places in each of the landward piers is to be 46 feet 6 inches and 46 feet respectively.

### MASONRY.

The masonry of the abutments and piers is to be of squared work. Each course is to be of the same thickness throughout, and, after the stones are set, the course is to be carefully brought to one level before proceeding with the next course. To suit the quarried materials, the courses above the foundations may vary from 12 to 16 inches in thickness, the face-work being composed of a course of headers and stretchers alternately. The header stones are to have bond into the wall of not less than 3 feet 6 inches, and are not to be less than 2 feet in length of outward face. The stretchers are not to be less than 2 feet in breadth on the bed, and 3 feet 6 inches on the outward face. These courses are to be laid so

as to break joint properly, as will be hereinafter more particularly described.

### HORIZONTAL ARCHES OF ABUTMENTS.

The back or landward side of the abutments is to be built in the form of a horizontal arch of an elliptical form, which is to spring from the wing walls. These walls are to be built in regular courses, forming a continuation of the masonry of the abutments. The stones of the exterior faces or soffits of these arches are to be laid in courses of header and stretcher alternately, the headers not being under 3 feet in length, and 1 foot 9 inches in breadth of bed, and the stretchers not under 1 foot 6 inches in breadth of bed, and 3 feet in length of face. They are to be worked in the beds and joints with chisel-draughts round the edges, and pick-dressed between. The horizontal arches are to set off from the foundation course as already described, and to be carried upwards by offsets in common with the abutments. Above these offsets the same arched form is to be carried up perpendicularly to the level of the top of the springing course, where there are 3 more offsets of one foot each. Above this there is a course of flag-stones at the same height as the covering stones of the spandril walls, afterwards to be described. The stones of the horizontal arches are to average 2 feet 6 inches in depth of bed at the springing of the arch, and 2 feet at the crown. They are not to be less than 2 feet 6 inches in length, to be built in vertical bond, and

closely jointed with the hearting of the abutment. The bed joints are to be dressed in a similar manner to those of the face-work of the wing walls, the outward face to be chisel-draughted round the edges for accuracy in setting, but the space between to be left undressed. The other dimensions to be taken from the plans.

### WING WALLS.

The beds, joints, and face-work of the wing walls are to be dressed in a similar manner to those of the horizontal arches. The hearting to be similar to that of the abutments. The dimensions of the different parts are to be taken from the written dimensions on the plan. In the opening behind the horizontal arches, and for 10 feet beyond the ends and sides of each of the wing walls, excavated matters from the foundations, mixed with gravel and a proper proportion of clay to give tenacity, are to be filled in to the top of the wing walls. This mixture is to be laid down in strata corresponding as nearly as may be with the courses of the building, each stratum being well beaten down as the courses are laid.

### DRESSING OF STONES.

The stones are all to be hewn or worked to the full size specified. The beds droved round the outward edges to the breadth of 3 inches, and broached with mallet and iron within these draughts. The outward faces of the courses below the level of summer water mark are to have $1\frac{1}{2}$ inch chisel-draughts round the

edges, and are to be pick and hammer-dressed between; above the summer water mark the face-work of the abutments and piers is to be neatly broached, and the horizontal joints chamfered to the breadth of 1 inch on the beds and faces. The hearting stones of the abutments and piers are, in like manner, to have $1\frac{1}{2}$ inch chisel-draughts round the edges of the horizontal beds and are to be broached between, but the vertical joints are to be dressed square with the pick or hammer. The impost courses are to be 15 inches thick, worked with a fillet of 3 inches in thickness, and a torus of 12 inches, projecting 8 inches beyond the face-work, as represented in the drawings. The cutwaters of the abutments and piers are to be built in courses along and in correspondence with those of the main body of the work, and finished with a continuation of the impost, surmounted by a blocking course of 2 feet 3 inches in thickness, formed for the support of the columns, as shown in the drawings.

The springing course of the arches is to be 4 feet in thickness. On the piers it is to be laid in three rows of stone; the two outer rows forming the springing stones are to be worked off in the face so as to form part of a voussoir having a soffit of 9 inches in breadth, and a bearing or breadth of bed on the pier of 3 feet 6 inches; and no stone is to be less than 2 feet 6 inches in length of face. The middle or closing row of the springing course is to be laid in two courses of such a breadth as to fill exactly the space left between the springing stones.

The springing course of the abutments is to be in all

respects similar to that of the piers, and the space between it and the back of the horizontal arch is to be filled up with square jointed hearting stones in two courses, so as to fill up the space exactly. The whole of the stones of those courses are also to be carefully worked in their bearing joints, and, when laid, are to be grouted with mortar.

The beds of the voussoirs are to be droved round the edge to the depth of 3 inches, and broached between these draughts. The end joints to be chisel-draughted, and broached off between; the face-work of the soffit of the arch is to be neatly broached, and the bed joints across the arches, and the heads of the ring courses, are to be chamfered to the breadth and depth of one inch on each side of the face or soffit. The heads of the stones of the ring courses are to project $1\frac{1}{2}$ inch beyond the retaining walls, as shown in the drawings. The back or crown of the voussoirs is to be worked off in the curve of the extrados to the breadth of 18 inches, so as to form a seat for the Aisler work of the retaining walls.

### CENTRE FRAMES.

Trussed centre frames, for building not fewer than three arches, are to be properly framed and constructed of Memel timber, and elm, or other hard wood, thoroughly connected with mortise and tenon joints, screw bolts, and iron straps, agreeably to a model to be prepared under the direction of the engineer. These centre frames are to be placed at distances not exceeding 4 feet apart, properly supported on beams

resting wholly on the scarcements of the abutments and piers, and covered with planks of 3 inches in thickness, on which the arches are to be turned. It is in the power of the engineer, however, to direct that five sets of centre frames, or one for each arch, shall be prepared for building the bridge; the contractor being paid for such extra centres.

### ARCHES.

The five arches of the bridge are to be segments of a circle of 65 feet radius. The bottom of the several springing courses is to be upon one level, and to be 9 feet 6 inches above the summer water mark. The span of the middle arch is to be 79 feet, and its rise or versed sine 13 feet 6 inches. The two arches next to the middle one are to be each 74 feet 6 inches, and their rise 11 feet 9 inches, and the remaining two arches next the abutments are to be each 65 feet in the span, and to rise 8 feet 9 inches.

The width of the bridge, measuring across the soffit of the arches, as before noticed, is to be 38 feet. The voussoirs or arch-stones are to measure not less than 3 feet in length across the soffit,—their depth at the key-stone or crown is to be 3 feet 6 inches, increasing as they approach the springing courses, according to a radius of curvature of 70 feet for the extrados, making the depth of the voussoir at the springing of the central arch 4 feet 6 inches, and that of the other arches proportionally less at the springing course, according to the above radius of curvature.

### RETAINING WALLS.

The retaining or outward spandril walls on each side of the bridge are to extend in a longitudinal direction $1\frac{1}{2}$ inch within the heads of the ring course, to which they are to be neatly fitted, and carried to the height of the curve coinciding with the crown of the arches. They are to be of the thickness of 3 feet at their foundation on the top of the springing courses, and to diminish to 2 feet 6 inches at the top. At the height of 7 inches under the range of the curve of the arches, a scarcement or offset of 9 inches in breadth is to be formed, as a rest for the covering pavement of the interior spandril walls.

The face-work of the masonry of the retaining walls is to consist of regular courses of Aisler work, neatly broached, laid header and stretcher. The headers are not to be less in the face than 2 feet in length, nor more apart than 9 feet in each course, nor shorter than the full thickness of the wall where they are laid. The stretchers are not to be less than 3 feet in length, nor 1 foot 4 inches in breadth. They are to be dressed square with mallet and iron on the beds, with a droved draught round the edge of $1\frac{1}{2}$ inch in breadth, and broached between, and the ends dressed square at least 6 inches back. The courses of Aisler work are to range from 15 to 12 inches in thickness, becoming gradually thinner toward the top of the walls; each course is to be chamfered in the face on the lower and upper beds, of the same dimensions as those specified for the piers. At the junction of the face courses with the face courses of

the arches, the stones are to be worked to a circular form, and cut square at the joint to the depth of 3 inches, so as to give them the requisite strength ; the corresponding stone of the succeeding course locking into that immediately below. The backing of the retaining walls is to be of coursed hammer-dressed masonry, corresponding in thickness with the Aisler course, and properly bonded with it. Circular holes of 3 inches in diameter are to be cut through the retaining walls communicating with the voids between the arches, in positions to be afterwards pointed out, for the admission of air, and the escape of any water which may percolate through the roadway of the bridge.

### COLUMNS AND PILASTERS.

Over the projections on each side of the abutments there is to be placed one square pilaster, and on each of the piers two columns executed in a plain Grecian Doric style. The columns on the piers of the middle arch are to measure 13 feet in height, including the capital, and 2 feet 4 inches in diameter at the base. Those on the two landward or outer piers are to be 10 feet in height, including the capital, and 2 feet 2 inches in diameter at the base. The pilasters on the abutments are to be 7 feet in height, including the capital, and 2 feet 4 inches by 2 feet. The columns and pilasters are to be set with their face line at the base 2 feet beyond the retaining wall. They are to be built in courses not exceeding 2 feet in thickness, and properly bonded with the Aisler of the retaining wall. The columns and

capitals are to be wrought agreeably to an enlarged drawing, which will be given to the contractor. The architrave on the columns is to be 1 foot 6 inches in depth, worked with a torus or band $1\frac{1}{4}$ inch in depth, and projecting one half inch from the face of the architrave; the breadth of which is to be made up of two stones, each of them being the full length and depth of the architrave. The frieze over the architrave is to be 1 foot 6 inches in depth or thereby at the lower end, corresponding with the frieze of the general elevation: this frieze is also to be of one stone, 1 foot in breadth, with closers to fill up the returns at the ends. It is to be backed with square dressed stones, laid as headers, extending to the inward face of the retaining wall, and equal in height to the frieze. Between the columns, on each of the centre piers, a semicircular niche is to be formed, 4 feet in width, and 10 feet 6 inches in height; the retaining wall at those parts being thickened behind to admit of the niche. The columns, architrave, and frieze are to be chisel-draughted in the beds to the breadth of 3 inches, and broached off with mallet and iron between. The face-work of the whole, including the soffit of the architrave, the niche, and the whole space between the columns, is to be executed in the best style of droved work, set in mortar, lipped to the breadth of 3 inches with Roman cement, and carried up along with the face-work and bonded with it.

### SPANDRIL WALLS.

Spandril walls are to be built between the outward

walls, parallel to them and butting against the horizontal arches of the abutment at either end of the bridge, and dividing the whole included space into eight equal compartments. These walls are to be 2 feet in thickness at the bottom, diminishing to 1 foot 6 inches at the top, which is to be at the level of 14 inches under the top line of the retaining walls. After being levelled, the spandril walls are to be coped with a course of hammer-dressed flag-stones, 1 foot 6 inches broad and 7 inches thick.

### TYE-WALLS.

Upon each of the piers a tye-wall, of 3 feet in thickness throughout, is to be built across the bridge, and is to be carried up and bonded with the retaining walls. These spandril and tye-walls are also to be properly bonded with each other, and built flush in mortar: holes 6 inches square are to be formed in each of the interior spandril and tye-walls, to preserve the circulation of air and drainage of moisture.

### FRIEZE.

The frieze course, which is to be laid on the top of the retaining walls, as above described, is to be 1 foot 6 inches in height, and to project 2 inches over the face of the retaining wall. The stones of which the frieze is to be built are not to be less than 3 feet 6 inches in length of face, 1 foot 6 inches in height, and 1 foot 2 inches in breadth of bed. They are to be worked with a droved draught of 5 inches in breadth in the lower bed, and 3 inches in the upper bed along the edges

next the face, and with a common chisel-draught round the other edges of the bed: the space between is to be broached, the end joints to be cut square to the depth of 6 inches, and the face-work neatly droved; the whole to be set in mortar and lipped with oil putty. The frieze course is to be backed with good rubble masonry, to the depth of 2 feet 8 inches from the face, and set in mortar.

### CORNICE.

The cornice, 1 foot 2 inches thick, and projecting 1 foot $3\frac{1}{2}$ inches from the frieze, is to be returned round all the projections over the columns and pilasters. The cornice is to be moulded agreeably to the enlarged diagram on the plan: no stone in it is to be under 3 feet 6 inches in length, or 3 feet in breadth on the lower bed, so as to have a bearing on the frieze course of not less than 1 foot $8\frac{1}{2}$ inches. The tails of the stones are to be checked down on the upper bed to a depth sufficient to admit the footpaths at their proper level. The stones are to be chisel-draughted round the beds to the breadth of 2 inches, and broached off between the draughts: the end joints are to be neatly droved to the depth of 1 foot 9 inches from the extreme projection of the moulding, and the whole of the moulded part is to be neatly droved. The cornice is to be set in mortar, and the projecting parts of the joints carefully grouted and pointed with oil putty.

### COVERING PAVEMENT.

Over each of the eight spaces between the spandril

walls, flag-stones, or covering pavement, in a hammer-dressed state, are to be set flush in mortar, for supporting the carriage roadway and footpaths. These flag-stones are each to contain not fewer than 8 superficial feet of rock, and are to be 7 inches in thickness, laid with a bearing of not less than 6 inches on the retaining walls, and 9 inches on the spandril walls. On this pavement, to the full width of the bridge between the retaining walls, clay puddle is to be laid to the depth of 12 inches, in strata 6 inches thick, which is again to be covered with a stratum of gravel, free of earthy particles, to the depth of 4 inches in the middle, diminishing at the sides to 2 inches.

### PARAPET WALLS.

The parapet walls on each side of the bridge are to be built of solid parapet Aisler masonry; the two parapet walls to be equal and similar to each other in all respects. The basement course is to be laid on the cornice course. It is to be 1 foot $2\frac{1}{2}$ inches in thickness, 12 inches in depth, and the stones are not to be less than 4 feet in length. The dado of the parapet is to be 12 inches thick, built in two courses, each 13 inches in height, the stones of which are to be laid in lengths of not less than 4 feet, and set in such a manner that the base may project one inch toward the outside. The coping course is to be 11 inches in thickness, and 1 foot 2 inches in breadth, projecting both outwards and inwards similar to the basement course, and laid in lengths of not less than 4 feet, the whole height of the parapet wall being 4 feet 1 inch from the upper surface

of the cornice course. The parapet walls are to be enlarged to the extent of the respective projections of the entablature, both in length and in breadth, and their base, dado, and cope, are respectively to be formed round these projections. These breaks or blockings are to be built with square dressed stones, corresponding with the several courses of the parapet wall. The stones throughout these walls are to be worked in the beds similar to those of the cornice and frieze. The cope and base are to be neatly droved, and the dado finely and closely broached. The coping of the parapet walls is to be chamfered on the upper bed from the middle line down to 9 inches from its lower bed on both sides. The stones of the coping are to be secured to each other by a cast-iron dowel $1\frac{1}{2}$ inch in diameter and 6 inches in length, let into each of the vertical joints; to be properly grouted and run up with Roman cement.

### LAMP IRONS.

Six lamp irons, formed agreeably to the enlarged drawings, are to be erected on each of the parapet walls of the bridge, and are to be connected with a train of gas pipes communicating with Adelphi and Clyde Streets. These lamp irons and gas pipes are to be provided at the expense of the trustees, but are to be fitted up at the cost of the contractor, agreeably to the directions of the engineer.

### FOOTPATHS.

A footpath of 5 feet in breadth, including the curb-

stone, is to be formed on each side of the bridge. For this purpose curb-stones are to be laid at the distance of 5 feet from the basement course of the parapet walls, the upper surface of the curb being 1 inch under the level of the said basement. These curb-stones are to be 1 foot 6 inches in depth, 9 inches in thickness at bottom, sloped to 6 inches at the top, and laid in lengths of not less than 2 feet. The spaces between the parapet walls and curb-stones, on either side, are to be laid with pavement on a solid bed of concrete, consisting of screened gravel mixed with one-fifth part of lime. The pavement to be neatly chisel-draughted round the edges, and the vertical joints squared, lipped, and pointed with Roman cement. The stones of the footpath are to measure not less than 2 feet in breadth, 4 feet 6 inches in length, and 6 inches in thickness, laid in mortar flush with the curb-stones at the one edge, and on a level with the bottom of the parapet walls on the other.

### DRAINS.

Along the under side of the curb-stones, on each side of the bridge, and 9 inches under their upper edge, sky-drains or gutters of Whinstone rock are to be formed. The stones in these drains are to be not less than 1 foot 6 inches in length, 1 foot 2 inches in breadth, and 9 inches in depth, having the groove or channel wrought to a segment of a circle, 7 inches in width by $2\frac{1}{2}$ inches in depth. By these gutters the drainage water is to be carried off the roadway into sewers communicating with the river.

### ROADWAY.

The carriage roadway is to be metalled to the full extent between the sky-drains and along the whole length of the bridge, and formed with a line of draught rising on the one side and falling on the other at the average rate of one perpendicular to thirty horizontal, with a transverse section rising 3 inches in the middle. The road is to be laid to the depth of 12 inches with broken Whinstone, each piece of which, in its largest dimensions, is to pass through a ring of $2\frac{1}{4}$ inches in diameter. The whole to be consolidated by being repeatedly passed over by a rolling machine of not less than $1\frac{1}{2}$ ton in weight.

### DESCRIPTION OF MASONRY.

The face or outside work of this bridge is to be built of Aisler masonry, as has been already more particularly specified and described. The stones are to be laid throughout the whole works on their natural beds, or as they lie in the quarry. The arch-stones are to break joint or overlap each other not less than 1 foot. In all other parts of the face-work the vertical or perpendicular joints of the one course are to overlap the joints of the course immediately below not less than 9 inches. The whole of the masonry of the bridge is to be set flush in mortar, prepared as described under that article. The joints of the other parts of the masonry are to be carefully grouted with mortar, and the face-work neatly pointed. The beds of the face-work of the abutments and piers, to the height of the springing course, are to

be lipped or laid to the depth of 3 inches with Roman cement, and the beds of the face-work of the retaining walls, the pilasters with their basements and entablature, the cornice, parapet walls, and other exposed parts of the work, are to be lipped to the same depth with oil putty prepared with a mixture of fine sand and tinted of a stone colour, to the satisfaction of the resident engineer.

MORTAR.

The mortar for the foundations of the work and abutments, with their wing walls, the piers and arches, the retaining, spandril, and parapet walls, and other parts of the bridge, is to be composed of well burned limestone, of quality approven of by the engineer. It is to be burned at or near the works, and slaked as it comes from the kiln with pure river water; it is then to be mixed with sand, free of earthy particles, in the proportion of one measure of lime to two measures of clean sharp sand, or fine screened gravel suitable to the species of work for which it is to be used. This mixture is then to be worked to a proper consistency by manual labour, or a horse-gin and mortar-rollers if required. But, notwithstanding the proportions of lime and sand being above specified, these are to be subject to alteration on the inspection of the mortar by the resident engineer, who is also to decide upon the quality of the Roman cement and oil putty for lipping and pointing the works.

QUARRIES.

The stones for the outward walls or face-work, in-

cluding the shafts of the piers and face-work of the abutments, wing walls, arch-stones, retaining and parapet walls, including the frieze and cornice of the bridge, are to be taken from the best and hardest rock of such of the following quarries as may ultimately be approven of by the engineer as being capable of producing stones of sufficient dimensions, and of proper quality: viz. Garscube, Balgray and Woodside, in the vicinity of Glasgow, Nitshill in Renfrewshire, Carbrook in Stirlingshire, or Humbie in Linlithgowshire. For all interior work, such as the hearting of the abutments and piers, backing of the wing walls and rubble work in general, together with the covering pavement, the stones may be taken from the best and hardest rock of any of the quarries above specified; but if it shall meet the convenience or views of the contractor to take the whole of the materials from the quarry or quarries ultimately approved of for the face-work by the engineer, the contractor shall be at liberty to do so, and the whole of the materials may thus come to be procured from such quarry or quarries, provided always, that it be without detention or disadvantage to the works: a provision which the contractor will keep in view in any engagement he may be disposed to enter into with the quarry masters.

The contractor must keep a mason in the principal quarry from which the materials are to be supplied, whose duty it shall be to make careful selection of the stones; but independently of this condition, no stone will be permitted to be laid in any of the abutments,

piers, arches, or in any part of the face-work of the bridge, having what are called *drys* or *cutters*, or any unsoundness in its general aspect or appearance; or which from colour or want of hardness may appear doubtful to the resident engineer or said chief engineer, who may cause such faulty stone or stones to be removed, though actually built into the work.

The curb-stones, sky-drains, and road metal shall be taken from one of the quarries in the neighbourhood of Glasgow, which shall be approven of by the engineer; and the foot pavement from Castle-hill quarry in Caithnesshire, or any others of approved quality.

### PROGRESS OF THE WORKS.

By the term of Martinmas, in the year 1828, the contractor is to complete the north abutment, with its wing walls, and three of the piers, to the level of the springing courses. By the term of Martinmas, 1829, the contractor is to complete the remaining abutment and pier, to turn the five arches of the bridge, and carry the retaining, spandril, and tye-walls to the height of 5 feet above the springing course: and by the term of Lammas, 1830, he is to complete and finish the whole of the remaining works of the bridge, hereinbefore specified and represented in the plans and drawings, signed by Robert Stevenson as chief engineer, and by the contracting parties as relative hereto.

### WORK-YARD.

The contractor shall enclose the work-yard to be

provided by the trustees for their works, extending to two acres or thereby, with a wooden paling: he is also to erect a work-yard lodge, consisting of two floors, agreeably to a plan approved of by the engineer. The ground-floor is to be for the use of the contractor, and the upper-floor for the use of the resident engineer and trustees. When the bridge works are completed, the contractor is to remove the said paling and work-yard lodge, (the materials being his property,) and to restore and leave the ground as nearly as may be in its present state, by the removal of all rubbish and chips of stones, to the satisfaction of the trustees.

### GENERAL OBLIGATIONS ON THE PART OF THE CONTRACTOR.

The contractor is to provide and pay for all timber and building materials, and for their carriage; he is also to pay the wages of all artificers and labourers employed in the work,—to provide a service bridge to facilitate his operations,—to furnish the pile-work for the foundations,—and provide three trussed centres for building the arches, with all their necessary supports and apparatus: he is also to furnish pumps, engines, cranes, tools, utensils, and implements of what kind or description soever as may be found necessary for preparing the foundations of the bridge, and executing and completing the whole superstructure, including its roadway and approaches, to the full extent of the bridge and its wing walls, in a safe, effective, substantial and workmanlike manner, under the direction and to the satisfaction of

Robert Stevenson, civil engineer; failing whom, to the satisfaction of any other civil engineer appointed by the trustees. The whole works are to be fully and completely executed at the sight of a resident engineer or inspector, who shall be approven and recommended by the chief engineer.

The contractor for these works shall not be at liberty to sublet any part of them without the consent of the chief engineer, excepting in so far as regards the works of excavation, quarrying stones, or the carriage of materials. The contractor shall find security for the due execution of the work and completion of the contract to the satisfaction of the trustees.

### GENERAL OBLIGATIONS ON THE PART OF THE TRUSTEES.

The trustees are to provide the land for the site of the bridge, a work-yard of not less than two acres, as contiguous to the site of the bridge as may be found expedient, with the necessary access to the works, over and above the contract price for the bridge: the said trustees are to pay the contractor the sum of £ for enclosing the work-yard, and upon all building materials and implements laid down for the purposes of the work they shall have a right of property. The trustees shall also pay the contractor £ toward the erection of a work-yard lodge. They shall also convey to the contractor all the privileges of their Act or Acts of Parliament in regard to the conveying of materials for the work, free of toll or turnpike dues; they shall like-

wise pay the salary of the resident engineer, and cause the contract price for the bridge to be paid to the principal contractor, or his order, by instalments, proceeding upon the certificates of the engineer, or agreeably to his directions, in manner following: viz.

One-twentieth part of the said contract price when a quantity of materials of the value of £100 sterling are laid down upon the ground, and the deed of contract signed by the contractor.

One-twentieth part further when each of the platforms of the abutments and piers are respectively laid.

One-twentieth part when the first abutment is brought up to the springing course.

One-fortieth part for each of the piers and remaining abutments, when they are brought to the height of the springing course.

One-fortieth part when the arch-stones of each of the five arches have been worked, or when a quantity of arch-stones equal to each arch are hewn, and ready for being set.

One-fortieth part when each of the centre frames for the middle and two of the side arches shall have been prepared, and are ready for the work.

One-twentieth part when the retaining, spandril, and tye-walls shall have been carried to the full height.

One-twentieth part when the covering stones, frieze, and cornice, are laid.

One-twentieth part when the parapet walls and roadway of the bridge are completed. And the balance remaining of the contract price when all the works are

completed, and the bridge finally taken off the hands of the contractor by a letter under the hands of the engineer.

Should it so happen that the foundations or any other part of the works of the bridge are required to be carried to a greater depth and extent than is herein described and represented in the accompanying plans and sections of the work, the contractor, upon the certificate of the engineer, shall be allowed an additional sum in proportion to the extent of extra work done. But it is expressly declared and understood that any such extra works are to proceed only on the previous orders given under the hands of the engineer, or by his order, otherwise such extra charge shall not be sustained. If, on the other hand, any of the foundations or other works shall be diminished in extent, a rateable and proportionable deduction shall be made on the contract price, agreeably to the schedule and rates for additional and short works filled up by the contractor, with specific prices, and presented by him along with his offer for the contract, to which that schedule is hereby declared to form a relative paper.

It is understood that the trustees shall not be bound to accept the lowest offerer, but shall have the choice of the contractor from among the offerers.

### CONCLUDING CLAUSE.

Should any misunderstanding arise as to the meaning and import of the foregoing specification or relative drawings, or about the quality and dimensions of mate-

rials, or the due and proper execution of the works, the same shall be explained and decided by the said chief engineer, whose award in all cases shall be final and binding upon the contracting parties. It is further declared and provided for, that if it shall at any time appear to the said chief engineer that the works are not proceeding in a proper and satisfactory manner agreeably to the contract, he shall have the power of discharging the contractor, taking the contract out of his hands, and employing others to finish the works. In the event of its being necessary to resort to such measures, the trustees shall pay such sums as shall be awarded by the engineer to the contractor for work done; but if it shall appear that the balance of the contract price is not sufficient for completing the remainder of the works, the difference, so ascertained and determined, shall be made good by the contractor; it being the full meaning and understanding of the contracting parties that the agreed upon contract price shall complete the whole works herein specified, and that the annexed schedule of prices refers only to extra works and deductions, proceeding upon such written orders for their execution as may be issued from time to time under the hands of the engineer.

When the works shall have been found by the engineer to have been duly executed and completed in all respects agreeably to the contract, and to his entire satisfaction, the same shall be intimated by a writing under his hands, within six months after the date of the same having been so declared. He shall then take the bridge

finally off the hands of the contractor, if the same shall then be found by him to be in good and substantial order. The balance of the contract price shall then be paid, and all further responsibility relative to the works removed from the contractor.

*Schedule of rates of prices, including labour and materials, for ascertaining the value of extra and short works, or of alterations, referred to in the foregoing specification of Hutcheson Bridge, Glasgow, by Robert Stevenson, Civil Engineer.*

| | £. | s. | d. |
|---|---|---|---|
| 1. Excavating foundations above the level of summer water mark, including removal, not exceeding 50 yards, shall be valued at per cubic yard | | | |
| 2. Excavating or dredging foundations under the level of summer water mark, including removal, not exceeding 50 yards, at per cubic yard | | | |
| 3. Memel timber or red pitch pine, in guage piles, independently of its value as old or used timber, at per cubic foot | | | |
| 4. Sheeting piles, in like manner independently of the old timber, at per cubic foot | | | |
| 5. Beech or Memel timber in bearing piles, sills, wale-pieces, and planking, at per cubic foot | | | |
| 6. Driving bearing piles close, after the manner of sheeting piles, at per lineal foot | | | |
| 7. Fitting and fixing wale-pieces and sills, at per lineal foot | | | |
| 8. Iron-work in screw bolts, spikes, and straps, at per lb. | | | |
| 9. Clay puddle in coffer-dams, ramming behind walls, and for roadway, at per cubic yard | | | |
| 10. Upfilling with broken Whinstone between the heads of bearing piles, at per cubic yard | | | |
| 11. Coursed pitching of cut freestone laid between the sills and foundations, at per cubic foot | | | |
| 12. Masonry in face-work and hearting of the abutments | | | |

£. s. d.

and piers under the level of summer water mark, at per cubic foot

13. Masonry in face-work and hearting of the wing walls, at per cubic foot
14. Masonry in face-work of abutments and piers from the level of summer water mark to the top of the impost course, at per cubic foot
15. Masonry in hearting of abutments and piers from the level of summer water mark to the top of the impost course, at per cubic foot
16. Masonry in springing courses of abutments and piers, at per cubic foot
17. Masonry in arches, at per cubic foot
18. Broached and chamfered Aisler work, at per superficial foot
19. Coursed rubbled masonry, at per cubic foot
20. Good common hammer-dressed rubble masonry, at per cubic foot
21. Pilasters built in courses, at per cubic foot
22. Frieze over pilasters, at per cubic foot
23. Cornice of bridge, at per lineal foot
24. Parapet walls of bridge, at per lineal foot
25. Curb-stones for foot pavement, at per lineal foot
26. Sky-drains for roadway, at per lineal foot
27. Foot pavement, at per superficial yard
28. Embanking approaches, stuff removed not exceeding 300 yards, at per cubic yard
29. Metalled roadway, at per cubic yard
30. Retaining walls, at per do.
31. Retaining walls with coursed face-work and rubble backing, at per cubic yard
32. Removal of excavated stuff per hundred yards, exceeding 300 yards, and not exceeding 600 yards, at per cubic yard
33. Hammer-dressed cope for spandril walls, at per lineal foot
34. Hammer-dressed pavement, at per superficial yard
35. Centre frames of arches, and work connected therewith, at per cubic foot
36. Built close drains measuring 2 feet 6 inches square, at per cubic foot

£. s. d.

37. If the width of the bridge shall be increased, and the roadway made broader than is described or shown in the plan and foregoing specification, the whole charge for extra work, including labour, materials, and implements of all kinds rendered necessary on this account for the work, shall not for every additional lineal foot of breadth upon the whole length of the bridge by which the roadway and approaches are so increased exceed the rate or sum of

38. And so in like manner and proportion for any diminution of breadth on the whole length of the roadway and approaches of the bridge, from that which is described or shown in the plan and foregoing specification, a rateable and corresponding deduction shall be made per foot of breadth, at the rate of

39. If the foundation of either of the abutments of the bridge, measuring from the point towards the river landward to the points or springing of the horizontal arch, shall be sunk or carried to a greater depth than is described or shown in the plan or foregoing specification, the whole charge for extra work, including labour, materials, and implements of all kinds rendered necessary for the work on that account, shall not for every additional vertical foot by which the depth is increased exceed the rate or sum of

40. And so in like manner and proportion for any diminution of depth, as aforesaid, by which the foundation of either of the abutments of the bridge is lessened in the specified depth, a rateable and corresponding deduction shall be made per foot perpendicular at the rate of

41. If the foundation of any of the piers of the bridge shall be sunk or lowered to a greater depth than is described and shown in the plan and foregoing specification, the whole charge for extra work, including labour, materials, and implements of all kinds rendered necessary for the work, on this account, shall not for every additional perpendicular foot by which the depth is increased exceed the rate or sum of

42. And so in like manner and proportion for any diminution of depth, as aforesaid, by which the foundations

£. s. d.

of any of the piers of the bridge are lessened in the specified depth, a rateable deduction shall be made per vertical foot at the rate of

43. If instead of the coupled pilasters, with their basements, entablatures, and connecting works, the retaining walls, cornice, and parapet walls be continued and extended throughout the elevation of the bridge in a plain form, such alteration will form a deduction upon the contract price, amounting to the sum of

*Note.*—All charges for extra work specified in the above schedule for any operation or article referable to labour or materials necessary for protecting works of this kind, though not enumerated and expressed, are understood to be provided for and covered in the charges respectively specified in the schedule, or otherwise left to the determination of the engineer.

(Signed and dated by the Offerer.)

# SUPPLEMENT

TO VOL. I.

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## THE MATHEMATICAL PRINCIPLES

OF

## MR. DREDGE'S SUSPENSION BRIDGE.

# THE MATHEMATICAL PRINCIPLES

OF

# MR. DREDGE'S SUSPENSION BRIDGE.

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A NOVEL and elegant principle has lately been adopted in the construction of *Suspension Bridges*,—a principle which promises to be of the greatest utility and importance, both as regards the quantity of material employed, and the stability and strength which it confers upon the structure.

One of the distinguishing features of this very ingenious invention, which is due to Mr. DREDGE, of Bath, consists in making the chains of sufficient magnitude and strength at the points of suspension, to support with safety the greatest permanent and contingent load, to which, under the circumstances of locality, they are ever likely to be exposed; and from thence, to taper or diminish them gradually to the middle of the bridge, where the strain becomes essentially evanescent.

The gradual diminution of the chains, however, is not the only circumstance which characterizes this mode of construction, and marks its superiority over the methods employed by other engineers. The suspending rods or bars that support the platform or road-way, instead of being hung vertically or

at right angles to the plane of the horizon, are inclined to it in angles which vary in magnitude from the abutments to the middle of the bridge, where the obliquity, as well as the stress upon the chains, attains its minimum value.

The principle developed by this obliquity of the suspending rods is singularly beautiful; but much judicious management is necessary on the part of the engineer, to fix upon that degree of obliquity which shall produce the greatest effect. Each bar is considered to perform its part in supporting the load, in proportion to its distance from the abutment, drawn into the sine of the angle of its direction, so that the entire series of suspending bars, transmits the same tension to the points of support as would be transmitted by a single bar reaching from thence to the middle of the bridge.

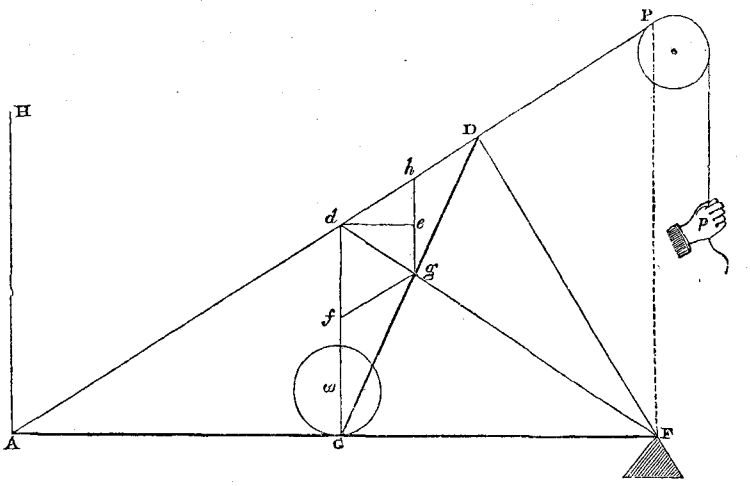
If this individual bar be considered as a straight inflexible line, the principles of calculation are identical with those of a lever when sustained in equilibrio by a single force applied at the remote extremity, and acting in the direction of the sustaining bar. But when the bar is curved or polygonal, as it necessarily must be in the case of a chain employed in the actual construction of a bridge, the process becomes a little more complicated, since it requires all the composant forces to be referred to the several portions of the chain as resultants; we mean such portions of it as are comprehended between any two contiguous bars. In every other respect, the principles which regulate the calculations are the same; and for this reason, it will be sufficient to establish the theory in reference to the straight line only, since the reader who is familiar with the *Composition* and *Resolution* of *Forces*, will find no difficulty in extending the process to a curve, and he who is not, will derive but little benefit from a perusal of the subject which is now to be laid before him.

Nothing can be more simple than the doctrine of the lever,

when sustained in equilibrio by a single force applied at a given point, and acting in a given direction: this is explained in every treatise on the principles of mechanical science, and is consequently within the reach of every individual connected with that or kindred pursuits: but a more important case of the problem remains to be considered, viz., that in which the equilibrium is maintained by a system of forces applied at different points in the length of the lever, and acting in different directions with respect to the horizon. This is the case that becomes assimilated to a suspension bridge, when the chains and suspending rods are simultaneously called into action in such a manner, that their joint effects shall be competent to maintain the system in a state of balanced rest.

In the case of a single force applied at the remote extremity of the lever, and having its direction inclined to the horizon in a given angle, while the load is applied at the middle of the length, and acting in the direction of gravity. Let A F (fig. 1)

Fig. 1.




be a lever, having one extremity resting on the fulcrum at F, and conceive it to be sustained in equilibrio by a power or force *p* applied at A the other extremity, and acting in the direction A P.

Conceive the lever, thus circumstanced, to be a heavy and

perfectly inflexible bar, of uniform figure and density in all its parts, and suppose the whole of its weight to be accumulated in G the centre of gravity; it is then reduced to the case of a straight inflexible lever void of weight, and having a load of given magnitude applied at its middle point.

From the fulcrum F draw the straight line F D perpendicular to A P, the direction of the sustaining force; then, by the principles of plane trigonometry, we have the following proportion, viz.,

$$\text{rad.} : \sin. < \text{F A D} :: \text{F A} : \text{F D}.$$

Or, by reducing the analogy and making radius = 1, it is

$$\text{F D} = \text{F A} \cdot \sin. < \text{F A D}.$$

Now, F D is the perpendicular distance between the fulcrum and the direction in which the force acts; consequently, by the property of the lever,

*The power acting in the direction* A P, *is to the weight acting in the direction of gravity, as the distance* F G *is to the distance* F D. *That is, the power and the weight are reciprocally as the distances* F G *and* F D.

Let $p$ = the magnitude of the force which is applied at A, and acting in the direction A P,

$d_n$ = A F, the whole length of the lever, or the distance from the fulcrum of the point at which the force acts,

$\phi_n$ = F A P, the angle of direction, or that contained between the plane of the horizon and A P the direction of the force,

$w$ = the weight of the lever or platform, supposed to be concentrated at G the middle of its length,

$\delta = \frac{1}{2} d_n$ = F G, the distance between the fulcrum and the point where the weight acts.

Then, by the property of the lever just enunciated, the state of

equilibrium in terms of our notation is expressed by the following analogy, viz.,

$$p : w : : \delta : d \underset{n}{\sin.}\ \underset{n}{\phi}.$$

Therefore, by multiplying extremes and means, the *equation of condition* is

$$p\, d \underset{n}{\sin.}\ \underset{n}{\phi} = \delta\, w. \qquad \text{(A)}$$

And by division, the value of the suspending power becomes

$$p = \frac{\delta}{d}\,.\, w \operatorname{cosec.} \underset{n}{\phi}. \qquad \text{(B)}$$

Now, by the principles of mechanics, the effect must obviously be the same at whatever point in the line A P the force may be applied; so that, if A P be a cord or chain supporting the weight $w$ by its tension, the measure of that tension is the same at every point, provided that its inclination in respect of the horizon is constantly the same. And if P be a point vertically situated with respect to the fulcrum F, let the cord or chain A P be suspended at P, in such a manner as to be kept perfectly tight by the strain which is excited in maintaining the equilibrium; then the value of the strain upon the point of suspension at P is expressed by equation (B).

Reverting to the equation of condition, and substituting $\frac{1}{2} \underset{n}{d}$ for $\delta$, the expression for the state of equilibrium becomes

$$p \underset{n}{\sin.}\ \underset{n}{\phi} = \tfrac{1}{2} w, \qquad \text{(C)}$$

or, by division, we obtain for the equilibrating power,

$$p = \tfrac{1}{2} w \operatorname{cosec.} \underset{n}{\phi}. \qquad \text{(D)}$$

This expression for the measure of the power which maintains the system in a state of rest is exceedingly simple; but being limited to the case where the weight acts at the middle of the lever, and the sustaining power at the remote extremity, it will be more advantageous in our subsequent inquiries to retain

the general value, as expressed above, in terms of the respective distances $d$ and $\delta$; because, in as far as the lever is concerned, the symbol $\delta$ may partake of all values less than $d$, the entire length of the bar on which the forces act.

If the power $p$, instead of acting in the direction A P oblique to the horizon, should act in the direction A H perpendicular to it, then $< \mathrm{F\,A\,H} = \phi = 90°$, and cosec. $\phi = 1$; in which case we have

$$p = \tfrac{1}{2} w. \qquad \text{(E)}$$

This is manifest; for one half the weight is sustained by the fulcrum, and the other half by the power applied at A.

In what precedes, we have established the equation of condition by reference to the principles of the lever only; but the same thing can readily be accomplished by the *Resolution* of *Forces*, and by the same means we at once obtain the magnitude of the oblique strain upon the fulcrum or abutment, and also the horizontal thrust. Through the point G (fig. 1) draw the straight line G $d$ at right angles to A F, and meeting A P the direction of the power in $d$. Draw $d$ F and D G, and from the point $d$ on the vertical line $d$ G set off $df$ at pleasure, to denote the magnitude of the weight $w$ acting in the direction of gravity. Upon $df$ complete the parallelogram $dfgh$, and $fg$ or $dh$ will represent the magnitude of the force in the direction A P, while the diagonal $dg$ denotes the oblique pressure on the fulcrum, or the magnitude of the force in the direction $d$ F.

Since the angles A D F and A G $d$ are right angles by construction, the quadrilateral figure F D $d$ G can be inscribed in a circle, and because the angles F D G and F $d$ G are subtended by the same chord F G, they are equal between themselves; the angles D F G and $dfg$ are also equal, each of them being equal to the complement of the given angle P A F. Con-

sequently, the triangles F D G and $f d g$ are similar, and their homologous or corresponding sides proportional; so that

$$\text{F D} : f d :: \text{F G} : f g.$$

Now F D, as we have already seen, is equal to F A . sin. $< \text{F A D} = d_n$ sin. $\phi_n$, and $f d = w$, F G and $f g$ being respectively represented by the symbols $\delta$ and $p$; hence we get

$$d_n \text{ sin. } \phi_n : w :: \delta : p,$$

from which, by equating the products of the extremes and means, we obtain

$$p\, d_n \text{ sin. } \phi_n = \delta w \text{ . (the same as equation A).}$$

In the plane triangle $d f g$ we have given the two sides $d f$ and $f g$, with the contained angle $d f g$, to find the third side $d g$; and for this purpose, a well known theorem in plane trigonometry gives

$$d g = \left\{ w^2 + p^2 - 2\, w p \text{ sin. } \phi_n \right\}^{\frac{1}{2}} \qquad \text{(F)}$$

But, according to equation (B), the value of $p$ in terms of $d_n$, $\delta$, $w$ and $\phi_n$ is $\frac{\delta}{d_n} . w$ cosec. $\phi_n$; and consequently, its square is $\frac{\delta^2}{d_n^2} . w^2$ cosec.$^2$ $\phi_n$; therefore, by substituting these values for $p$ and $p^2$ within the brackets, we get

$$d g = w \sqrt{1 + \frac{\delta^2}{d_n^2} \text{ cosec.}^2 \phi_n - \frac{2\,\delta}{d_n} \text{ cosec. } \phi_n \text{ sin. } \phi_n}.$$

By the principles of trigonometry, the cosecant of any arc or angle to radius unity, is the same as the reciprocal of the sine to the same radius; hence we have cosec. $\phi_n$ sin. $\phi_n = 1$, so that the expression for the oblique thrust becomes

$$r = \frac{w}{d_n} \sqrt{d_n^2 - 2\, d_n \delta + \delta^2 \text{ cosec.}^2 \phi_n}. \qquad \text{(G)}$$

Where the symbol $r$ denotes the resultant of the weight or load $w$ and the oblique force $p$, and corresponds in magnitude and direction to the oblique thrust on the fulcrum at F.

There is still another quantity remaining to be determined, the magnitude of which may sometimes be required in a case of actual construction; we mean the horizontal thrust upon the abutment, occasioned by the joint effects of the weight and the oblique power, when exerting themselves with full intensity in their respective directions.

Let $dg$ (fig. 1), which represents the oblique thrust upon the abutment, be resolved into the two forces $de$ and $ge$, the one of them being parallel to the horizon, and the other perpendicular to it; then it is obvious that in so far as the thrust upon the abutment is concerned, the effect of the vertical force is wholly lost, that of the horizontal force only being employed in producing the strain.

Since, by construction, the straight line $de$ is parallel to A F, the angle $hde$ is equal to the angle $\mathrm{PAF} = \underset{n}{\phi}$, and we have already seen, equation (B), that $fg$ or its equal $dh$ is expressed by $\frac{\delta}{\underset{n}{d}} . w \operatorname{cosec.} \underset{n}{\phi}$; therefore by the principles of plane trigonometry we get

$$\text{rad.} : \frac{\delta}{\underset{n}{d}} . w \operatorname{cosec.} \underset{n}{\phi} : : \cos. \underset{n}{\phi} : de.$$

Therefore, if $h$ be put to denote the horizontal thrust upon the abutment, the reduction of the above proportion will give

$$h = \frac{\delta}{\underset{n}{d}} . w \operatorname{cosec.} \underset{n}{\phi} \cos. \underset{n}{\phi} .$$

But, according to the calculus of sines, the product of the cosine and cosecant of any arc or angle is equal to the cotangent; hence we have $\operatorname{cosec.} \underset{n}{\phi} \cos. \underset{n}{\phi} = \cot. \underset{n}{\phi}$, from which, by substitution, we get

$$h = \frac{\delta}{d}_{n} . w \cot. \phi_{n} . \qquad \text{(H)}$$

It therefore appears that the equations (B), (G), and (H), exhibit the several particulars required in and arising from the equilibration of a lever; and in order that these particulars may be clearly understood, we shall now proceed to show in what manner the equations are to be reduced, by applying them to the resolution of a numerical example.

EXAMPLE.—Suppose the length of a lever to be 56 feet, each foot in length being equal to a weight of 40·56 lbs. avoirdupoise; now, if one extremity of this lever rests upon a fulcrum, while it is sustained in equilibrio by a force applied at the other extremity, and acting in a direction which makes an angle of 32 degrees with the horizon; what is the magnitude of the sustaining force, and what are the thrusts excited upon the fulcrum, both in reference to the oblique and horizontal direction?

The first demand of this question is fulfilled by equation (B), the second by (G), and the third by (H); in all of which, the value of $\delta$ is 28 feet, being equal to one half the length of the lever. And since the system is placed under the same conditions as it would be, if a load equivalent to the weight of the lever were applied at its centre of gravity, the lever itself being then considered as without weight, it follows that

$$w = 56 \times 40{\cdot}56 = 2271{\cdot}36 \text{ lbs.}$$

Since the angle of direction of the suspending power is 32 degrees, its natural cosecant is 1·88708, and the numerical expression for the magnitude of the power by which the equilibrium is maintained is

$$p = \frac{28 \times 2271{\cdot}36 \times 1{\cdot}88708}{56} = 2143{\cdot}12 \text{ lbs.}$$

The logarithmic process for the reduction of this expression is as follows, viz.,

$\delta = 28$ feet . . . . . log. 1·4471580
$w = 2271{\cdot}36$ lbs. . . . . log. 3·3562859
$\phi_n = 32$ degrees . . . . . log. cosec. 0·2757903
$d_n = 56$ feet . . . . . ar. co. log. 8·2518120
} add.

Therefore the value of $p$, the equilibrating power,
is 2143·12 lbs. . . . . log. 3·3310462 sum.

Hence it appears, that a power of 2143·12 lbs., acting at the remote extremity of the lever, and having its direction inclined to the horizon in an angle of 32 degrees, will balance a load of 2271·36 lbs. applied at the middle of the length, and acting in the direction of gravity, the lever itself having no weight.

The data remaining as above, the numerical expression for the value of the oblique thrust on the abutment is

$$r = \frac{2271{\cdot}36}{56}\left\{56^2 - 2\times 56\times 28 + 28^2\times 1{\cdot}88708^2\right\}^{\frac{1}{2}}$$

By a slight examination of the quantities within the brackets, it will readily appear that the second or negative term is precisely equal to that which precedes it; consequently, they destroy one another, and the expression reduces to the following, viz.,

$$r = \frac{2271{\cdot}36\times 28\times 1{\cdot}88708}{56} = 2143{\cdot}12 \text{ lbs.}$$

From this we infer that when the load is applied at the middle of the lever, and the balancing power at the remote extremity, the magnitude of the oblique thrust on the fulcrum is precisely the same as the magnitude of the sustaining power. This is also manifest from the figure; for the triangle $g\,d\,h$ or $d\,g\,f$ is isosceles.

Finally, the data still remaining, the numerical expression for the magnitude of the horizontal thrust on the fulcrum becomes

$$h = \frac{28\times 2271{\cdot}36\times 1{\cdot}6003345}{56} = 1817{\cdot}47 \text{ lbs.}$$

Where 1·6003345 is the natural cotangent of 32 degrees; hence the following logarithmic process,

$\frac{\delta}{d} = \frac{28}{56}$ . . . . . log. 9·6989700
$n$

$w = 2271{\cdot}36$ . . . . . log. 3·3562859

$\phi = 32°$ . . . . . log. cot. 0·2042108
$n$

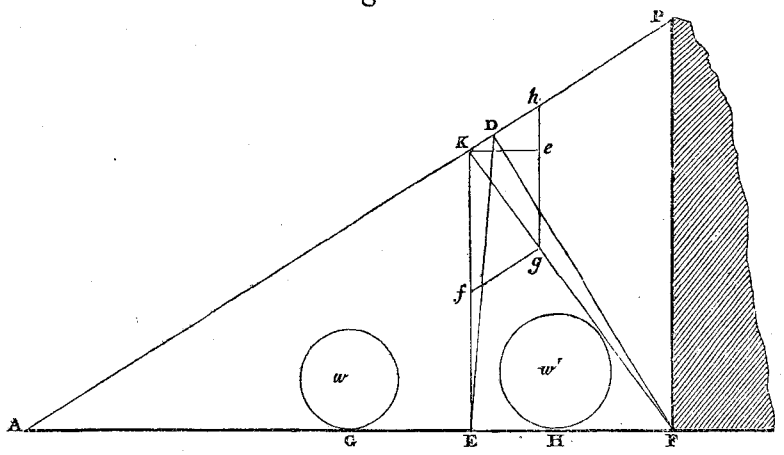
Therefore the value of $h$, the horizontal thrust, is

1817·47 lbs. . . . . log. 3·2594667 sum.

The preceding investigation has reference only to the weight of the lever or platform, which is constantly the same; but in the case of a bridge, this constancy of weight cannot obtain by reason of the transit loads to which it is exposed; and as the loads of transit are continually varying their distances from the fulcrum, and thereby producing different effects at every instant, it becomes necessary, in order to render our investigation complete, to show in what manner these effects are to be estimated, and incorporated with the effect produced by the weight of the platform.

Let $w$ (fig. 1[a]), denote the weight equivalent to that of the

Fig. 1[a].




platform, applied at G the centre of gravity or middle point of the lever, and let $w'$ be any other load applied at the point H.

Then it is manifest, from the principles of the lever, that the two weights $w$ and $w'$ will, according to their situations, perform the same office in equipoising the system, as would be performed by a single weight equivalent to their sum applied at E their common centre of gravity, and acting in a direction perpendicular to the horizon. Divide the distance G H into two parts G E and H E, which are to one another reciprocally as the weights applied at G and H; then is E the position of the load, which is equivalent to $w + w'$. Through the point E, and perpendicular to F A, draw the straight line E K, meeting A P in K; and let F D be perpendicular to A P. Draw D E and F K, and upon K E set off K $f$, equal to the compound load $w + w'$, and complete the parallelogram K $f g h$; then will $f g$ or K $h$ denote the force in direction A P, and K $g$ the oblique pressure on the fulcrum at F, arising from the conjoint action of the constant weight $w$ and the extraneous weight $w'$. Through the point K draw K $e$ at right angles to $h g$, and K $e$ will denote the horizontal thrust upon the fulcrum, produced by the conjoint action of the two forces already specified; and if the load $w'$ should move forward or backward to any other point of the lever, the same mode of construction will still apply, so that the principle is general, whether the load be constant or variable.

If the symbol $\delta$ denote the distance F E between the fulcrum F and the common centre of gravity of the two weights $w$ and $w'$, the other quantities remaining as before, then the equation of *condition* for the constant and variable load becomes

$$p \underset{n}{d} \sin. \underset{n}{\phi} = \delta (w + w'), \qquad (\mathrm{H}^a)$$

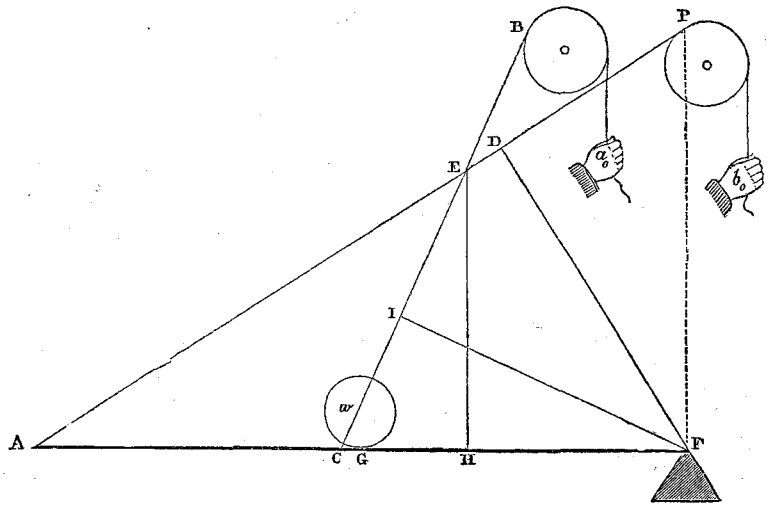
and the absolute value of the force in direction A P is

$$p = \frac{\delta}{\underset{n}{d}} . (w + w') \operatorname{cosec}. \underset{n}{\phi}. \qquad (\mathrm{H}^b)$$

In the next place, instead of the lever being sustained in

equilibrio by a single power $p$, applied at A and acting in the direction A P (fig. 2), let us suppose that the equilibrium is

Fig. 2.




produced by the joint effects of two forces $b_0$ and $a_0$, the one being applied at A the extremity of the lever, and acting in the direction A P, while the other is applied at some intermediate point of the lever, as C, and acting in the direction C B, the angles of inclination P A F and B C F being respectively denoted by the symbols $\phi_n$ and $\phi_0$, while the distances A F and C F are indicated by $d_n$ and $d_0$, the symbol $\delta$ being the same as in the preceding case; namely, the distance between the fulcrum and the point where the weight acts; that is, the distance G F.

Let the point C, at which the subsidiary force $a_0$ is applied, lie between the points A and G; that is, between the middle of the lever and its extreme point at A; then, since the magnitude of the weight $w$ and its distance from the fulcrum are the same as before, its effect upon the lever will manifestly be the same also, and consequently the joint effects of the two forces $b_0$ and $a_0$, which produce the equilibrium in this case, will

be equivalent to the effect of the single force $p$, by which it was previously produced.

But by the laws of mechanics the effect of any force which acts upon a lever at a given distance from the fulcrum, and in a given direction,—

*Is expressed by the magnitude of the given force, drawn into the distance between the fulcrum and the point at which it acts, and again into the sine of the angle of its direction.*

We have already seen, equation (A), that the effect of a single force which balances the weight of the lever is expressed by $p\,d \sin.\,\phi$; this, therefore, is the quantity to which the effects of the forces $b$ and $a$, now balancing the weight, must be equal, and with which they must jointly be compared.

Now, by the property of the lever as above enunciated, the effect of the force $b$ applied at A, and acting in the direction A P, is $b\,d \sin.\,\phi$, the distance A F and angle P A F being the same as before. And in like manner the effect of the force $a$ applied at the point C, and acting in the direction C B, is $a\,d \sin.\,\phi$; consequently, by taking the sum of these effects, the equation of condition or equilibrium in the case of two sustaining forces becomes

$$a\,d \sin.\,\phi + b\,d \sin.\,\phi = p\,d \sin.\,\phi.$$

But by equation (A) we have $p\,d \sin.\,\phi = \delta\,w$; consequently, by substitution, we obtain

$$a\,d \sin.\,\phi + b\,d \sin.\,\phi = \delta\,w. \qquad \text{(I)}$$

The composition of this equation is very simple; for from the fulcrum F draw the straight lines F D and F I respectively perpendicular to A P and C B, the direction in which the forces

act. Then, since $AF = d_n$ and $CF = d_0$, while the angles PAF and BCF are respectively denoted by the symbols $\phi_n$ and $\phi_0$, it follows from the principles of plane trigonometry, that

$$FD = d_n \sin.\ \phi_n\ ,\ FI = d_0 \sin.\ \phi_0\ .$$

And multiplying these distances by the magnitudes of the respective powers, the products will indicate the mechanical effects, and the sum of those effects must be equal to the magnitude of the weight $w$ drawn into the distance FG; and this equality constitutes the equation of *condition* or *equilibrium* given above.

If we transpose the first term of equation (I), the effect of the force $b_0$ becomes

$$b_0 d_n \sin.\ \phi_n = \delta w - a_0 d_0 \sin.\ \phi_0\ ,$$

and by division we obtain

$$b_0 = \frac{\delta w - a_0 d_0 \sin.\ \phi_0\ ,}{d_n \sin.\ \phi_n}.$$

But, according to the calculus of sines, the sine and cosecant of any arc or angle to radius unity are reciprocally equal; consequently, a more elegant expression for the magnitude of $b_0$ is that which follows, viz.,

$$b_0 = \frac{\left(\delta w - a_0 d_0 \sin.\ \phi_0\right) \text{cosec.}\ \phi_n}{d_n}. \qquad \text{(K)}$$

This supposes the magnitude of the force $a_0$, as well as all the other quantities composing the equation except $b_0$, to be known; but if $b_0$ be the given force, the equation expressing the value of $a_0$ becomes

$$a_0 = \frac{\left(\delta w - b_0 d_n \sin.\ \phi_n\right) \text{cosec.}\ \phi_0}{d_0}. \qquad \text{(L)}$$

If the equations (B) and (K) be compared with each other, it will readily appear that the former exceeds the latter by the quantity $\frac{a\ d\ \text{sin.}\ \phi}{d} \times \text{cosec.}\ \phi$; for it is obvious that

$$p - b = \frac{\delta}{d} \cdot w \,\text{cosec.}\ \phi - \frac{\left(\delta w - a\ d\ \text{sin.}\ \phi\right)\text{cosec.}\ \phi}{d} = \frac{a\ d\ \text{sin.}\ \phi\ \text{cosec.}\ \phi}{d}.$$

We therefore infer, that in the case of a single subsidiary force $a$, as in fig. 2, the tension upon the cord or chain A P is less than the tension in fig. 1, by the quantity

$$\frac{a\ d\ \text{sin.}\ \phi\ \text{cosec.}\ \phi}{d},$$

so that the diameter or section of the cord or chain may be diminished in the same proportion, the force in the direction C B diminishing the strain to that extent.

If the force $a$, which is applied at the point C, have its direction perpendicular to the length of the lever A F; then sin. $\phi = 1$, and the effect in that case is simply $a\ d$ instead of $a\ d\ \text{sin.}\ \phi$, and the corresponding value of $b$ is therefore

$$b = \frac{\left(\delta w - a\ d\right)\text{cosec.}\ \phi}{d}. \qquad \text{(M)}$$

If the equations (B) and (M) be now compared with each other, it will be seen that the former exceeds the latter by the quantity $\frac{a\ d\ \text{cosec.}\ \phi}{d}$; for, by subtraction, it is

$$p - b = \frac{\delta}{d} \cdot w \,\text{cosec.}\ \phi - \frac{\left(\delta w - a\ d\right)\text{cosec.}\ \phi}{d} = \frac{a\ d\ \text{cosec.}\ \phi}{d}.$$

Now, when the angle of inclination $\phi$ is of any magnitude

whatever less than 90 degrees or a right angle, sin. $\phi_0$ is less than unity, and, consequently, this last remainder is greater than the former by the quantity $\frac{a_0 d_0 \text{ cosec. } \phi_n}{d_n}(1 - \text{sin. } \phi_0)$; for by subtraction we get

$$\frac{a_0 d_0 \text{ cosec. } \phi_n}{d_n} - \frac{a_0 d_0 \text{ sin. } \phi_o \text{ cosec. } \phi_n}{d_n} = \frac{a_0 d_0 \text{ cosec. } \phi_n}{d_n}(1 - \text{sin. } \phi_0).$$

From this we infer that when the direction of the force $a_0$ is perpendicular to the lever, the magnitude of the force $b_0$ in direction A P necessary to maintain the equilibrium, is less than when the direction of the force $a_0$ is oblique. This inference, at first sight, would seem to be at variance with the principles insisted on by Mr. Dredge; and it manifestly is so, if we conceive the vertical and oblique forces to be applied at the same point of the lever; but if we suppose them to be referred to the same point of the cord or chain A P, the case will be very different; for then the quantity indicated by the symbol $d_0$ diminishes, and the quantity of diminution is proportional to the cosine of the inclination or obliquity. For, draw the perpendicular E H, then C F is the length of the lever in the case of the oblique force in direction C E, and H F its length for the vertical force in direction H E; but C H = C F − H F, and C H = C E cos. $\phi_0$; for, by the principles of plane trigonometry, it is

$$\text{rad.} : \text{C E} : : \text{cos. } \phi_0 : \text{C H} = \text{C E cos. } \phi_0.$$

In this equation, however, the quantity C E is still unknown, but it may be determined in the following manner:

$$\text{A C} = \text{A F} - \text{C F} = d_n - d_o,$$

and by the property of the plane triangle, as demonstrated in the 32nd proposition of the first book of Euclid's Elements of Geometry, the angle E C F is equal to the sum of the angles E A C and A E C; therefore by transposition it is

$$A\,E\,C = \phi_0 - \phi_n,$$

and by trigonometry we have

$$\text{sin.}\,(\phi_0 - \phi_n) : d_n - d_0 :: \text{sin.}\,\phi_n : C\,E,$$

and by reducing the analogy it becomes

$$C\,E = \frac{(d_n - d_0)\,\text{sin.}\,\phi_n}{\text{sin.}\,(\phi_0 - \phi_n)}. \qquad (N)$$

Let this value of C E be substituted instead of it in the above value of C H, and we obtain

$$C\,H = \frac{(d_n - d_0)\,\text{sin.}\,\phi_n\,\text{cos.}\,\phi_0}{\text{sin.}\,(\phi_0 - \phi_n)}. \qquad (O)$$

By a well known theorem in the calculus of sines, we have

$$\text{sin.}\,(\phi_0 - \phi_n) = \text{sin.}\,\phi_0\,\text{cos.}\,\phi_n - \text{cos.}\,\phi_0\,\text{sin.}\,\phi_n;$$

therefore, by substitution, we get

$$C\,H = \frac{(d_n - d_0)\,\text{sin.}\,\phi_n\,\text{cos.}\,\phi_0}{\text{sin.}\,\phi_0\,\text{cos.}\,\phi_n - \text{cos.}\,\phi_0\,\text{sin.}\,\phi_n},$$

and by expunging the factor $\text{sin.}\,\phi_n\,\text{cos.}\,\phi_0$ from both terms of the fraction, it finally becomes

$$C\,H = \frac{d_n - d_0}{\text{tan.}\,\phi_0\,\text{cot.}\,\phi_n - 1}. \qquad (P)$$

If from C F the leverage for the oblique force in direction C B, we subtract the value of C H, as found above, the remainder H F will be the leverage for the vertical force in direction H E. Thus we have

$$\text{H F} = \frac{d \tan.\ \phi \cot.\ \phi - d}{\tan.\ \phi \cot.\ \phi - 1}. \qquad \text{(Q)}$$

When the value of the distance H F, as found above, is less, equal to, or greater than $\text{F I} = d \sin.\ \phi$, the effect of the vertical force will be less, equal to, or greater than that of the oblique force; by this we mean, that the magnitude of the force being the same in both cases, its effect in the vertical direction will be less, equal to, or greater than its effect in the oblique direction, according as H F is less, equal to, or greater than F I.

Since E is a fixed point through which the directions of both the vertical and oblique forces must pass, it follows that while the distance F H is constant, the distance F C may partake of all magnitudes between F A and F H, while the angle E C F may librate between the angles F H E and F A P; that is, it may vary from a right angle to $\phi$, which is its minimum limit.

From this view of the subject it is obvious that the perpendicular F I depends upon the magnitude of the angle E C F; and because the point C may move either towards A or H, while the straight line C B passes constantly through E, it follows, that the inclination admits of such a magnitude as to give the force in the direction C B a maximum effect; that is, when $d \sin.\ \phi$ is a maximum or the greatest possible.

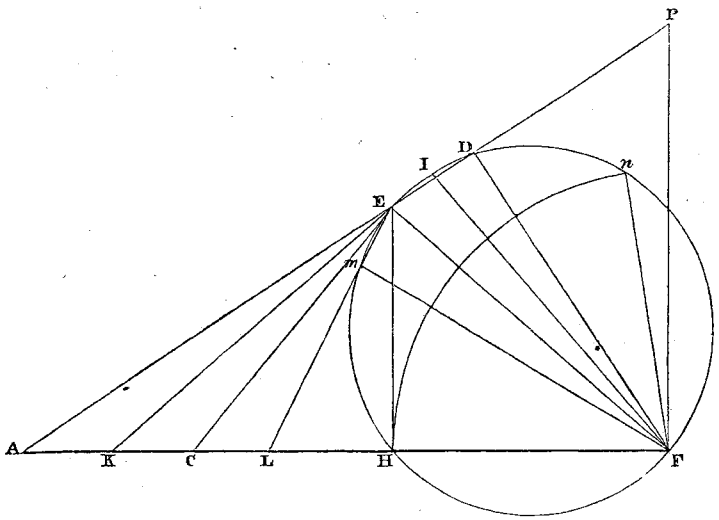
The method of determining the angle which gives to the oblique force its maximum effect is very simple; but if such a principle were adopted in the construction of suspension bridges, the results in many instances would differ very widely from the practice of the inventor, and especially as applied to the suspending rods in the *Victoria Bridge* across the Avon at Bath.

From drawings which we have seen of that structure, it would appear that the angles of direction diminish according to some law, as we recede from the abutments towards the centre

of the bridge; but in making the obliquity such as to give the force in any direction its maximum effect, the angles continually increase instead of decreasing, as will become manifest from the following constructions.

Let A F (fig. 3) be a lever resting on the fulcrum at F, and

Fig. 3.




sustained in equilibrio by a cord or chain applied at A, and fastened to a hook or some other immoveable object at P.

Let E be any point in the length of the cord, at which a subsidiary force is to be applied in order to lessen the strain upon A E, the lower portion thereof. The point C is required, so that C E being joined, a given force acting in the direction C E may produce a maximum effect.

Draw the straight line F E to connect the fulcrum with the given point, and through the point E, draw E C at right angles to F E, and meeting the lever A F in C; then is C E the direction of the force required. Upon F E, as a diameter, describe the circle F H E, intersecting the lever in the point H, and draw E H; then is E H perpendicular to the lever in H, so that H E would be the direction of the vertical force passing through E the given point.

If the point C be supposed to move towards A in such a

manner that C E may become K E; then it is manifest that K E produced will intersect the circumference of the circle again in I; draw F I, and F I will be perpendicular to K E, the direction of the given force; this is obvious, for the angle in a semicircle is a right angle. Now F I is manifestly less than F E, for F E is the diameter of the circle F H E, and by the property of the circle, the diameter is the greatest chord that can be inscribed in it. When the point K coincides with A, the point I coincides with D, in which case $F D = \underset{n}{d} \sin. \underset{n}{\phi}$.

Again, suppose the point C to move towards H until it arrives at the point L; then L E being drawn will necessarily intersect the circumference of the circle in the point *m*. Draw F *m*, which, by the property of the circle, will be perpendicular to L E, the direction of the given force supposed to be applied at L. Now, since F E is the diameter of the circle F H E, and F *m* any chord in it not passing through the centre, F E is greater than F *m*; but it has also been shown to be greater than F I; and since the effect of the force in the directions K E, C E, and L E, is represented by the perpendiculars F I, F E, and F *m*, it follows, that the effect in the direction C E is a maximum; hence the truth of the construction is manifest. When the point L coincides with H, the straight line L E coincides with H E, and H E becomes the direction of the vertical force, the effect of which is represented by F H.

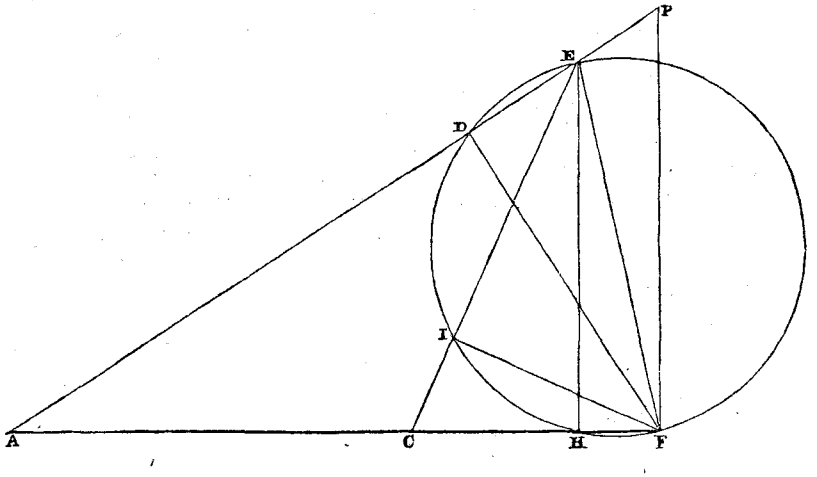
Since H and D are the points in which the circle is intersected by the lever A F, and the cord or chain A P, it follows, that the effect of the given force may be represented by all the chords that can be drawn in the circle between the points D and H, F D being the minimum value when the point C moves towards A, and F H the minimum value when it moves in the opposite direction. And all these values will manifestly be greater than F H; for from the centre F with the distance F H, describe the arc H *n* intersecting the circumference of the

circle in $n$; then it is evident that all the chords originating at F, and meeting the circumference between the points $n$ and H, must be greater than F $n$ or F H. Thus far, therefore, there is a manifest advantage in causing the force to act in the oblique direction.

The preceding construction determines the point in the lever at which the oblique force must be applied to produce a maximum effect for all points in the cord or chain between A and D; but it does not apply to points on the opposite side of D towards P, for in that case the direction of the force would meet the lever on the opposite side of A, which is manifestly contrary to the conditions of the problem as applied to the construction of bridges; for all the suspending bars on one side are supposed to meet the platform between the abutment and the middle of the bridge. The fact is, the direction which gives the maximum value is the direction of the cord itself, the force being applied at the point A; this will be obvious from the subjoined construction.

Let A F (fig. 4) be the lever resting upon the fulcrum at F,

Fig. 4.




and A P the direction of the force, which, acting singly at A, maintains the equilibrium. Let C be the point at which the subsidiary force is applied, and C E the direction in which it

acts; then, *it is proposed to show, that the greatest effect will be produced by the given power when it acts in the direction* A E *coincident with the cord or chain* A P.

Draw the straight line F E, connecting the fulcrum with that point in the cord through which the direction of the force passes, and upon F E as a diameter describe the circle F H D E, intersecting the lever A F in the points F and H; the cord A P in the points D and E, and C E the direction of the force in the point I. Draw F I, F D, and E H, these will respectively be perpendicular to the lines C E, A P, and A F. Now, it is obvious, that if the point C moves along the lever A F, either towards A or H, the straight line C E will intersect the circle in points towards D or H accordingly, and the perpendicular F I will increase or diminish according as the intersections fall towards D or H.

When the point C coincides with A the extremity of the lever, the straight line C E will coincide with A E, the direction of the cord, and the perpendicular F D will then attain its maximum value; hence the truth of the proposition is manifest.

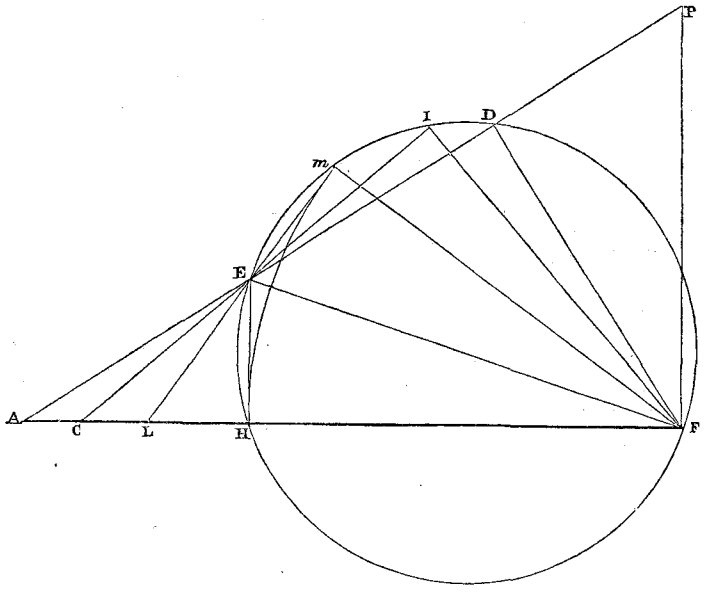
Again, when the point C coincides with the point H, the straight line C E will coincide with the perpendicular H E; and in that case the value of F H is a minimum; from which we infer that in all cases where the direction of the subsidiary force intersects the cord between the points D and P, the effect is the least possible when the force is exerted in a vertical direction.

In both the preceding constructions the position of the point E has been so assumed as to give the force, when acting in an oblique direction, a greater effect than when it acts in a vertical direction; and lest the inattentive reader should imagine that this is universally the case, we shall, in what immediately follows, endeavour to show that the effect of the

force acting vertically, may also be either equal to or greater than the effect of the same force acting obliquely, when the directions in both cases pass through the same point of the cord.

Let A F (fig. 5) be the lever, similarly situated as in all

Fig. 5.




the preceding cases, A P being the direction of the single equilibrating power. Take any point E in A P, and let the point E thus assumed be that through which the direction of the subsidiary force has to pass.

Draw F E, and upon F E as a diameter describe the circle F H E D, intersecting the lever in F and H, and the cord or chain A P in E and D. Let C be any point in the lever at which the subsidiary force is applied, and through the point E draw the straight line C E, meeting the circumference of the circle in I; then is C E the direction of the force, and F I perpendicular to C I, the representative of its effect. About the fulcrum F as a centre, with the distance F H describe the arc H *m* intersecting the circumference of the circle in *m*, and draw the straight lines F *m* and *m* E L; then is L E the direction of the oblique force acting at the point L, and the perpendicular

F $m$ is the representative of its effect. But F $m$ is equal to F H by the construction, and F H denotes the effect of the vertical force applied at H, and having its direction passing through E, the given point; therefore, in this case, the effects of the vertical and oblique forces are the same.

Now, it is manifest from the figure that the chord F $m$ is greater than any other chords that can be drawn from the fulcrum F to terminate in the arc $m$ D; therefore F $m$ is greater than F I; but F I is the representative of the effect produced by the oblique force acting at C in the direction C E, and F $m$, as we have just shown, denotes the effect of the vertical force acting at H in the direction H E; so that the effect of the oblique force may be either greater, equal to, or less than that of a vertical force of the same magnitude, supposing the directions in which they act to pass through the same point.

Having thus demonstrated the conditions that must be satisfied in order to produce a maximum effect, and having shown that the value of the oblique force may be either equal to, greater, or less than that of the vertical force whose direction passes through the same point of the cord, it now remains for us to show in what manner the several equations which we have investigated are to be reduced.

EXAMPLE.—Let the length of the lever, its weight, and the value of the angle $\phi$ remain, as in the example under equation (H), and suppose a subsidiary force $a$ to be applied at C at the distance of 26·5 feet from A the remote extremity of the lever, and having its direction passing through E at a distance of 44 feet from the point A (see fig. 2 preceding). It is required to determine the value of the force $b$, or the tension upon the lower portion of the cord A E, on the supposition that the required force acting in direction A E, produces the same effect in maintaining the equilibrium, as the force $a$ acting in direction C E.

$n$

$0$

$0$

$0$

The first step to be performed in the resolution of this example is to determine the angle $F C E = \phi_0$; and for this purpose we have given the length of the lever $F A = 56$ feet, and the angle $F A P = 32$ degrees; therefore, by the rules of plane trigonometry, we have

$$A P = 56 \text{ sec. } 32^\circ, \text{ and } F P = 56 \text{ tan. } 32^\circ.$$

These lines, however, are not necessary in the calculation; because, from the right-angled triangle A H E, the quantities A H and H E can be found directly from the data, without having recourse to similar triangles: thus we get

$$A H = 44 \text{ cos. } 32^\circ, \text{ and } H E = 44 \text{ sin. } 32^\circ;$$

the absolute numerical values being $44 \times \cdot 8480481 = 37 \cdot 3141164$, and $44 \times \cdot 5299193 = 23 \cdot 3164492$ feet respectively. Therefore, by subtraction, it is $C H = 37 \cdot 3141164 - 26 \cdot 5 = 10 \cdot 8141164$ feet; consequently, by trigonometry, we have

$$\frac{H E}{C H} = \frac{23 \cdot 3164492}{10 \cdot 8141164} = 2 \cdot 1561122 = \text{tan. } \phi_0 = \text{tan. } 65^\circ\ 7'\ 5''.$$

Having thus determined the angle of direction for the subsidiary force $a_0$, the equations by which the actual values of $a_0$ and $b_0$ are to be found are the following, viz.,

$$a_0 d_0 \text{ sin. } \phi_0 = b_0 d_n \text{ sin. } \phi_n,$$

$$\text{and } a_0 d_0 \text{ sin. } \phi_0 + b_0 d_n \text{ sin. } \phi_n = p_n d_n \text{ sin. } \phi_n = \delta w,$$

or, by substituting in this last equation the effect of the force $a_0$ in the first, we get

$$2 b_0 d_n \text{ sin. } \phi_n = \delta w.$$

Therefore, by restoring the numerical values of the several given quantities, the corresponding value of $b_0$ becomes

$$b_0 = \frac{28 \times 2271 \cdot 36}{2 \times 56 \times \cdot 5299193} = \frac{567 \cdot 84}{\cdot 5299193} = 1071 \cdot 56 \text{ lbs.}$$

Here, then, we have a strain of 1071·56 lbs. upon the lower

part of the cord; but by the example under equation (H), we found the strain throughout the cord with only one equilibrating force to be $p=2143{\cdot}12$ lbs., and this corresponds to the tension on the upper part in the present instance; hence it appears that by the application of the oblique force $a_0$, the lower part of the cord is relieved of a strain equal to $2143{\cdot}12-1071{\cdot}56=1071{\cdot}56$ lbs., being just half the original tension; and, consequently, the strength of the cord may be diminished in the same proportion. To find the magnitude of the force $a_0$ which acts in the direction C E, we must have recourse to the equation

$$a_0 d_0 \sin.\ \phi_0 = b_0 d_n \sin.\ \phi_n.$$

Now $d_0$, the distance between the fulcrum and the point in the lever where the force $a_0$ acts, is $d_0 = \text{A F} - \text{A C} = \text{C F} = 56 - 26{\cdot}5 = 29{\cdot}5$ feet; therefore, by substituting the several numerical values in the above equation, the value of $a_0$ becomes

$$a_0 = \frac{1071{\cdot}56 \times 56 \times {\cdot}5299193}{29{\cdot}5 \times {\cdot}9071767} = 1188{\cdot}23 \text{ lbs. nearly.}$$

Suppose, now, that the magnitude of the force $a_0$ and the angle of its direction are actually given by the question, instead of being determined by calculation as above, then the value of the force $b_0$ will be found as follows from equation (K), where we have

$$b_0 = \frac{(28 \times 2271{\cdot}36 - 1188{\cdot}23 \times 29{\cdot}5 \times {\cdot}9071767) \times 1{\cdot}8870799}{56} = 1071{\cdot}56 \text{ lbs.}$$

If the direction of the force $a_0$ forms with the lever an angle of 90°, then by equation (M) the value of $b_0$ becomes

$$b_0 = \frac{(28 \times 2271{\cdot}36 - 1188{\cdot}23 \times 18{\cdot}6858836) \times 1{\cdot}8870799}{56} = 1394{\cdot}92 \text{ lbs.}$$

The difference between the two values of $b_0$ found above is $1394{\cdot}92 - 1071{\cdot}56 = 323{\cdot}36$ lbs., from which the advantage of using the force $a_0$ in the oblique direction becomes manifest.

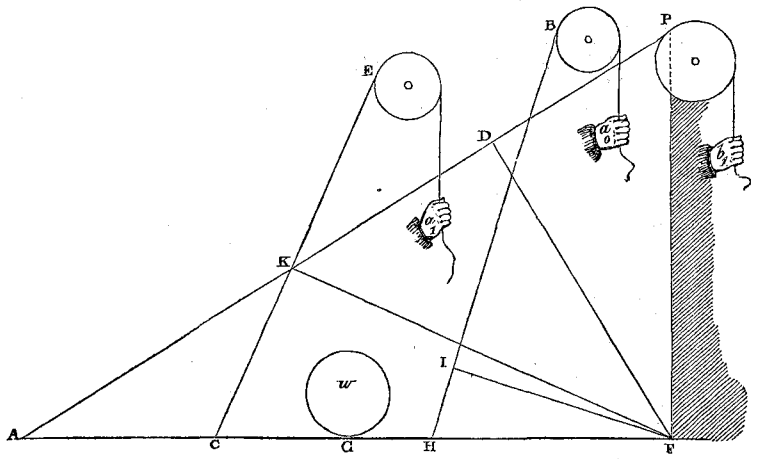
In the preceding process we have taken the value of $d_0$ $= 18{\cdot}6858836$ feet, being equal to the difference between the whole length of the lever F A and A H; that is, F H $= 56 - 37{\cdot}3141164 = 18{\cdot}6858836$; but the same thing may be determined directly from the data by equation (Q), on the supposition that the angle of direction when the force $a_0$ acts obliquely is known; that is, the value of the force $b_0$ corresponding to the vertical action of the force $a_0$ can be determined from the oblique action of $a_0$ without any previous calculation, by the following formula:

$$b_0 = \frac{\text{cosec. } \phi_n}{d_n} \left\{ \delta w - \frac{a_0 (d_0 \tan. \phi_0 \cot. \phi_n - d_n)}{\tan. \phi_0 \cot. \phi_n - 1} \right\}.$$

An equation which is very easily reduced by substituting the respective numbers as indicated by the combinations.

Let us now suppose that another subsidiary force as $a_1$ (fig. 6)

Fig. 6.




is added to the system, applied at the point C, and acting

in the direction C E. Then, since the weight of the lever remains the same, as well as the distance from the fulcrum of the point in which it is conceived to be concentrated, it will constantly require the same amount of power to effect the equilibrium, whatever may be the number of forces employed, in whatsoever manner they may be applied, and in whatever directions they may act.

Therefore, in the present case, with two subsidiary forces $a_0$ and $a_1$ acting in the directions H B and C E, and the third force $b_1$ acting in the direction A P, it will require the conjoint effects of all the three to maintain the equilibrium; that is, it will require the effect of the force $a_0$ in the direction H B, together with the effect of $a_1$ in the direction C E, and that of $b_1$ in the direction A P.

Now, by the notation already established, we have the angle $\mathrm{F\,H\,B} = \phi_0$ and the distance $\mathrm{F\,H} = d_0$, while the angle $\mathrm{F\,A\,P} = \phi_n$ and the distance $\mathrm{F\,A} = d_n$. In like manner let the angle F C E be denoted by $\phi_1$, and the distance F C by $d_1$; then, if from the fulcrum F the perpendiculars F I, F K, and F D be respectively drawn, and each of them multiplied by the magnitude of its corresponding force, the equation of *equilibrium* arising from their conjoint effects becomes

$$a_0 d_0 \sin.\ \phi_0 + a_1 d_1 \sin.\ \phi_1 + b_1 d_n \sin.\ \phi_n = \delta w. \qquad \text{(R)}$$

Therefore, by reducing this equation in respect of $b_1$ we obtain

$$b_1 = \frac{(\delta w - a_0 d_0 \sin.\ \phi_0 - a_1 d_1 \sin.\ \phi_1) \operatorname{cosec.}\ \phi_n}{d_n}. \qquad \text{(S)}$$

If the equations (K) and (S) be compared with each other, it will be seen that the former exceeds the latter by the quantity

$$\frac{a\ d\ \text{sin.}\ \phi\ \text{cosec.}\ \phi}{\underset{1}{1}\quad\underset{1}{1}\qquad\quad\underset{n}{1}\qquad\quad\underset{n}{n}},$$

for by subtraction we have

$$\underset{0}{b}-\underset{1}{b}=\frac{(\delta w-a\ d\ \text{sin.}\ \phi)\ \text{cosec.}\ \phi}{\underset{0}{a}\ \underset{0}{d}\qquad\underset{0}{0}\qquad\underset{n}{n}}-\frac{(\delta w-a\ d\ \text{sin.}\ \phi-a\ d\ \text{sin.}\ \phi)\ \text{cosec.}\ \phi}{\underset{0}{a}\ \underset{0}{d}\qquad\underset{0}{0}\quad\underset{1}{a}\ \underset{1}{d}\qquad\underset{1}{1}\qquad\underset{n}{n}}=\frac{a\ d\ \text{sin.}\ \phi\ \text{cosec.}\ \phi}{\underset{1}{a}\ \underset{1}{d}\qquad\underset{1}{1}\qquad\underset{n}{n}}.$$

Consequently, the tension on the cord or chain A P, when there are two subsidiary forces $\underset{0}{a}$ and $\underset{1}{a}$, is less than when there is only one subsidiary force, as $\underset{0}{a}$. The cord in the latter case may therefore be made smaller than it is in the former, for since there is less strain there is no necessity for so much strength.

If, instead of supposing the cords, to which the subsidiary forces are attached, to be brought over pulleys and stretched according to the intensity of their respective forces, let them be simply attached to the primary cord or chain A P, as represented in figs. 7 and 8 following; then, if they be stretched to the same degree by means of the load which they are employed to support, the nature of the action will be precisely the same as indicated in figs. 2 and 6, and the whole strain will be thrown upon the point of suspension at P, in the same

Fig. 7.

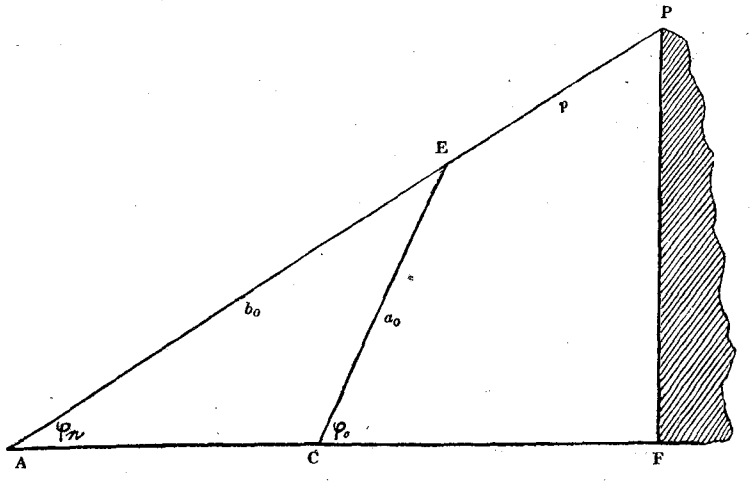
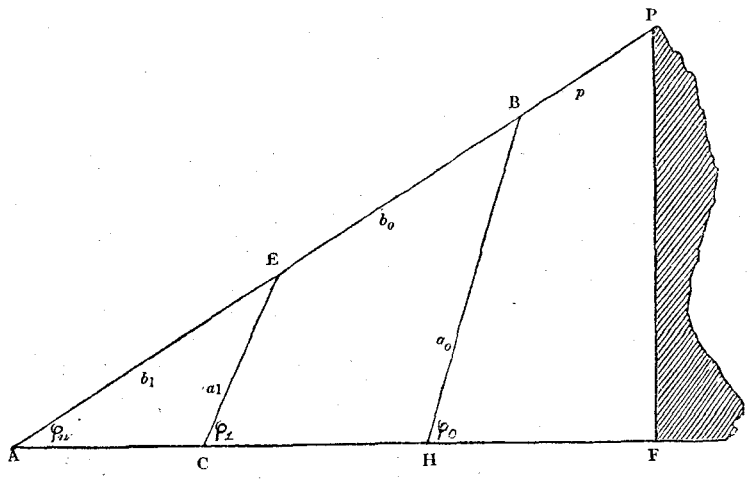


Fig. 8.




manner as if it were transmitted through the suspending bars of a bridge. From these two cases of subsidiary forces, as compared with each other, and also with the case of a single force, the law by which the tension on the cord or chain decreases becomes manifest; it is therefore unnecessary to pursue the investigation further, for by merely extending the notation the successive terms of the series can be obtained from each other by simple induction.

We shall give a tablet of the terms as far as seven subsidiary forces, which we presume will be amply sufficient to enable the attentive reader to detect the law by which the strain diminishes; and this law being once discovered, its application to the construction of suspension bridges becomes a matter of obvious utility. The series of equations denoting the state of equilibrium are as follow:

Force in the direction of the cord or chain . . . . . $p\,d\sin.\phi = \delta w$ ($n$, $n$)

Ditto with one subsidiary force . . . . . $a\,d\sin.\phi + b\,d\sin.\phi = \delta w$ ($0\,0$, $0$, $0\,n$, $n$)

Ditto with two subsidiary forces . . . . . $a\,d\sin.\phi + a\,d\sin.\phi + b\,d\sin.\phi = \delta w$ ($0\,0$, $0$, $1\,1$, $1$, $1\,n$, $n$)

Ditto with three subsidiary forces . . . . . $a\,d\sin.\phi + a\,d\sin.\phi + a\,d\sin.\phi + b\,d\sin.\phi = \delta w$ ($0\,0$, $0$, $1\,1$, $1$, $2\,2$, $2$, $2\,n$, $n$)

Ditto with four subsidiary forces . . . . . $a\,d\sin.\phi + a\,d\sin.\phi + a\,d\sin.\phi + a\,d\sin.\phi + b\,d\sin.\phi = \delta w$ ($0\,0$, $0$, $1\,1$, $1$, $2\,2$, $2$, $3\,3$, $3$, $3\,n$, $n$)

Ditto with five subsidiary forces . . . . . $a\,d\sin.\phi + a\,d\sin.\phi + a\,d\sin.\phi + a\,d\sin.\phi + a\,d\sin.\phi + b\,d\sin.\phi = \delta w$ ($0\,0$, $0$, $1\,1$, $1$, $2\,2$, $2$, $3\,3$, $3$, $4\,4$, $4$, $4\,n$, $n$)

Ditto with six subsidiary forces . . . . . $a\,d\sin.\phi + a\,d\sin.\phi + a\,d\sin.\phi + a\,d\sin.\phi + a\,d\sin.\phi + a\,d\sin.\phi + b\,d\sin.\phi = \delta w$ ($0\,0$, $0$, $1\,1$, $1$, $2\,2$, $2$, $3\,3$, $3$, $4\,4$, $4$, $5\,5$, $5$, $5\,n$, $n$)

Ditto with seven subsidiary forces $a\,d\sin.\phi + a\,d\sin.\phi + a\,d\sin.\phi + a\,d\sin.\phi + a\,d\sin.\phi + a\,d\sin.\phi + a\,d\sin.\phi + b\,d\sin.\phi = \delta w$ ($0\,0$, $0$, $1\,1$, $1$, $2\,2$, $2$, $3\,3$, $3$, $4\,4$, $4$, $5\,5$, $5$, $6\,6$, $6$, $6\,n$, $n$)

If we transpose this class of equations in such a manner that the terms involving the quantity $b$ with its several characteristics may stand alone on one side, the effects of the forces on the successive portions of the cord A P, estimated in order from the point of suspension, will become obvious. The transposed equations are as below, having 7 instead of $n$, that being its numerical value for a system of eight forces.

$$p_7 d_7 \sin.\phi_7 = \delta w$$

$$b_{07} d_7 \sin.\phi_7 = \delta w - a_{00} d_0 \sin.\phi_0$$

$$b_{17} d_7 \sin.\phi_7 = \delta w - \{a_{00} d_0 \sin.\phi_0 + a_{11} d_1 \sin.\phi_1\}$$

$$b_{27} d_7 \sin.\phi_7 = \delta w - \{a_{00} d_0 \sin.\phi_0 + a_{11} d_1 \sin.\phi_1 + a_{22} d_2 \sin.\phi_2\}$$

$$b_{37} d_7 \sin.\phi_7 = \delta w - \{a_{00} d_0 \sin.\phi_0 + a_{11} d_1 \sin.\phi_1 + a_{22} d_2 \sin.\phi_2 + a_{33} d_3 \sin.\phi_3\}$$

$$b_{47} d_7 \sin.\phi_7 = \delta w - \{a_{00} d_0 \sin.\phi_0 + a_{11} d_1 \sin.\phi_1 + a_{22} d_2 \sin.\phi_2 + a_{33} d_3 \sin.\phi_3 + a_{44} d_4 \sin.\phi_4\}$$

$$b_{57} d_7 \sin.\phi_7 = \delta w - \{a_{00} d_0 \sin.\phi_0 + a_{11} d_1 \sin.\phi_1 + a_{22} d_2 \sin.\phi_2 + a_{33} d_3 \sin.\phi_3 + a_{44} d_4 \sin.\phi_4 + a_{55} d_5 \sin.\phi_5\}$$

$$b_{67} d_7 \sin.\phi_7 = \delta w - \{a_{00} d_0 \sin.\phi_0 + a_{11} d_1 \sin.\phi_1 + a_{22} d_2 \sin.\phi_2 + a_{33} d_3 \sin.\phi_3 + a_{44} d_4 \sin.\phi_4 + a_{55} d_5 \sin.\phi_5 + a_{66} d_6 \sin.\phi_6\}$$

The foregoing may be considered as a mathematical illustration of the principle upon which the tapering of the chain depends; but the subject is still involved in difficulty as regards the distribution of the forces, and the position of the oblique suspending rods; for here there is no conditional equation to direct us, and on this account the successful application of the principle to practice, must in a great measure depend upon the sagacity and skill of the engineer by whom the fabric is raised.

In our humble opinion, a system of calculation may be instituted upon the principle of making the angles of direction vary by a constant second difference, at equidistant points along the platform, the limiting angles

being adapted to the circumstances of construction, which must of course be fixed upon before any calculation whatever can be made.

Another condition in the system of calculation might be, to have the effects of all the suspending forces equal; we do not mean the absolute magnitude of the forces, but the effects as referred to their particular directions, when compared with the state of equilibrium; this would manifestly give a series of equal differences for the tensions on the chain A P.

If the several equations above given be successively subtracted from one another, the series of corresponding remainders will stand as below, viz.:—

$$
\begin{aligned}
(p - b)_{0}\ d_{7} \sin.\ \phi_{7} &= a_{0}\ d_{0} \sin.\ \phi_{0} \\
(b - b)_{0\ 1}\ d_{7} \sin.\ \phi_{7} &= a_{1}\ d_{1} \sin.\ \phi_{1} \\
(b - b)_{1\ 2}\ d_{7} \sin.\ \phi_{7} &= a_{2}\ d_{2} \sin.\ \phi_{2} \\
(b - b)_{2\ 3}\ d_{7} \sin.\ \phi_{7} &= a_{3}\ d_{3} \sin.\ \phi_{3} \\
(b - b)_{3\ 4}\ d_{7} \sin.\ \phi_{7} &= a_{4}\ d_{4} \sin.\ \phi_{4} \\
(b - b)_{4\ 5}\ d_{7} \sin.\ \phi_{7} &= a_{5}\ d_{5} \sin.\ \phi_{5} \\
(b - b)_{5\ 6}\ d_{7} \sin.\ \phi_{7} &= a_{6}\ d_{6} \sin.\ \phi_{6}
\end{aligned}
\qquad \text{(T)}
$$

And if these equations be severally divided by the constant quantity $d_{7} \sin.\ \phi_{7}$, the actual values of the differences will be as follows:—

$$
p - b_{0} = \frac{a_{0}\ d_{0} \sin.\ \phi_{0} \operatorname{cosec.}\ \phi_{7}}{d_{7}}
$$

$$
b - b_{0\ 1} = \frac{a_{1}\ d_{1} \sin.\ \phi_{1} \operatorname{cosec.}\ \phi_{7}}{d_{7}}
$$

$$
b - b_{1\ 2} = \frac{a_{2}\ d_{2} \sin.\ \phi_{2} \operatorname{cosec.}\ \phi_{7}}{d_{7}}
$$

$$b_2 - b_3 = \frac{a\,d\,\sin.\,\phi\,\mathrm{cosec.}\,\phi}{d}{}^{3\ 3\qquad 3\qquad 7}_{7} \qquad \text{(U)}$$

$$b_3 - b_4 = \frac{a\,d\,\sin.\,\phi\,\mathrm{cosec.}\,\phi}{d}{}^{4\ 4\qquad 4\qquad 7}_{7}$$

$$b_4 - b_5 = \frac{a\,d\,\sin.\,\phi\,\mathrm{cosec.}\,\phi}{d}{}^{5\ 5\qquad 5\qquad 7}_{7}$$

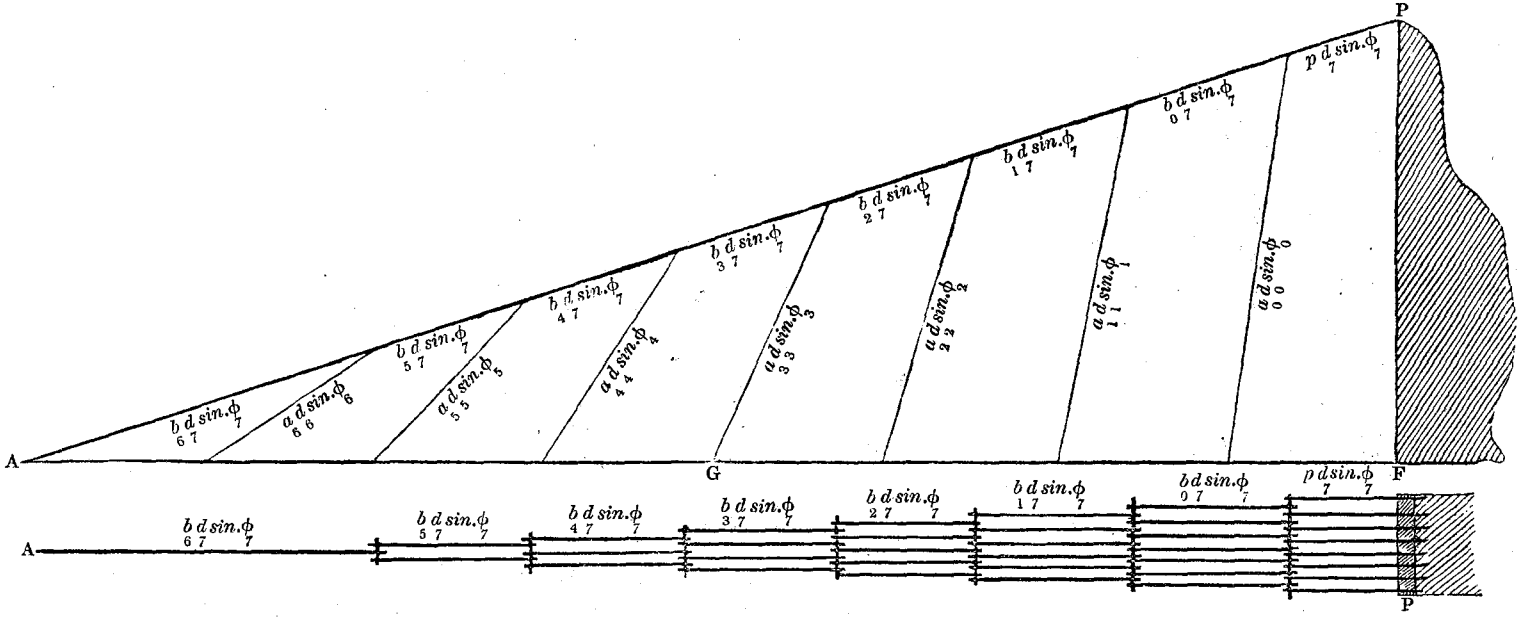
$$b_5 - b_6 = \frac{a\,d\,\sin.\,\phi\,\mathrm{cosec.}\,\phi}{d}{}^{6\ 6\qquad 6\qquad 7}_{7}$$

Consequently, the difference of the strains upon the first and last portions of the cord becomes

$$b_6 = \left\{ \frac{a\,d\,\sin.\,\phi + a\,d\,\sin.\,\phi + a\,d\,\sin.\,\phi + a\,d\,\sin.\,\phi + a\,d\,\sin.\,\phi + a\,d\,\sin.\,\phi + a\,d\,\sin.\,\phi}{d}{}^{0\ 0\quad 0\ 1\ 1\quad 1\ 2\ 2\quad 2\ 3\ 3\quad 3\ 4\ 4\quad 4\ 5\ 5\quad 5\ 6\ 6\quad 6}_{7} \right\} \mathrm{cosec.}\,\phi_7$$

The following drawing will represent the several quantities involved in the equations, and the tapering plan of the chain will indicate the manner in which the strain diminishes by the application of the subsidiary forces.

ELEVATION.




PLAN OF THE CHAIN.

The following numerical example may be useful in showing how a calculation is to be instituted, supposing that the conditions previously mentioned are admitted as being sufficient to guide the operations of practice in a case of actual construction.

EXAMPLE.—Let A F in the preceding figure be equal in length to 120 feet, and suppose it to be a straight inflexible lever entirely divested of weight and moveable about the fulcrum at F. Let the distance A F be divided into eight equal parts of 15 feet each, and at the several points of division let forces be applied, whose directions make with the horizon angles of 81, 78, 73, 66, 57, 46, 33, and 18 degrees respectively, estimated in order from the fulcrum. Now, suppose a load of 250 tons to be applied at G, the middle of the lever, and to be kept in equilibrio by the simultaneous action of all the forces; it is required to determine the strains upon the different portions of the hypothenuse A P, admitting the several oblique forces acting in their respective directions to produce the same or an equal effect.

Since the load which balances the accumulated effect of all the forces is 250 tons applied at a distance of 60 feet from the fulcrum, the conditions of equilibrium as referred to the property of the lever require that

$$15\,a_0 \sin. 81^\circ + 30\,a_1 \sin. 78^\circ + 45\,a_2 \sin. 73^\circ + 60\,a_3 \sin. 66^\circ + 75\,a_4 \sin. 57^\circ + 90\,a_5 \sin. 46^\circ$$
$$+ 105\,a_6 \sin. 33^\circ + 120\,b_7 \sin. 18^\circ = 250 \times 60 = 15000.$$

Now, the eighth part of this, or 1875 tons, is the effect of each of the forces in producing the equilibrium; consequently the actual magnitude of each of the forces can easily be found as follows:—

Beginning with the oblique force which is nearest to the fulcrum, and proceeding in order to the most remote, the actual magnitudes of the forces to produce equal effects, accord-

ing to the obliquity of their directions, will be respectively as below:—

$$a_0 = \frac{p\ d\ \sin.\ \phi_7}{8\ d_0\ \sin.\ \phi_0} = \frac{1875}{15\ \sin.\ 81^\circ} = 126{\cdot}5581 \text{ tons},$$

$$a_1 = \frac{p\ d\ \sin.\ \phi_7}{8\ d_1\ \sin.\ \phi_1} = \frac{1875}{30\ \sin.\ 78^\circ} = 63{\cdot}8963 \text{ tons},$$

$$a_2 = \frac{p\ d\ \sin.\ \phi_7}{8\ d_2\ \sin.\ \phi_2} = \frac{1875}{45\ \sin.\ 73^\circ} = 43{\cdot}5705 \text{ tons},$$

$$a_3 = \frac{p\ d\ \sin.\ \phi_7}{8\ d_3\ \sin.\ \phi_3} = \frac{1875}{60\ \sin.\ 66^\circ} = 34{\cdot}2074 \text{ tons}, \qquad \text{(V)}$$

$$a_4 = \frac{p\ d\ \sin.\ \phi_7}{8\ d_4\ \sin.\ \phi_4} = \frac{1875}{75\ \sin.\ 57^\circ} = 29{\cdot}8091 \text{ tons},$$

$$a_5 = \frac{p\ d\ \sin.\ \phi_7}{8\ d_5\ \sin.\ \phi_5} = \frac{1875}{90\ \sin.\ 46^\circ} = 28{\cdot}9617 \text{ tons},$$

$$a_6 = \frac{p\ d\ \sin.\ \phi_7}{8\ d_6\ \sin.\ \phi_6} = \frac{1875}{105\ \sin.\ 33^\circ} = 32{\cdot}7871 \text{ tons}.$$

These are the respective magnitudes of the subsidiary forces branching off from the primitive direction A P; and the corresponding magnitudes of the forces acting on the first and last portions of the hypothenuse or chain are as follows, viz.:—

$$p = \frac{\delta\ w}{d_7\ \sin.\ \phi_7} = \frac{15000}{120\ \sin.\ 18^\circ} = 404{\cdot}5086 \text{ tons},$$

$$b_6 = \frac{\delta\ w}{8\ d_7\ \sin.\ \phi_7} = \frac{1875}{120\ \sin.\ 18^\circ} = 50{\cdot}5636 \text{ tons}.$$

Having thus determined the magnitudes of all the subsidiary forces $a_0, a_1, a_2, a_3, a_4, a_5$, and $a_6$, together with the magnitudes of the forces which act on the first and last portions of the hypo-

thenuse or chain, it is easy to determine the magnitudes of the forces which act on all the intermediate portions; for, since by the conditions of construction, the effects of all the oblique forces are the same, it is obvious that all the equations in class (U) must be equal to the same constant quantity; that is, the forces acting on the different portions of the hypothenuse or chain, diminish in a regular progression from the highest to the lowest point, and the constant difference may be found from any one of the equations of the class just specified; or it may be found by taking the difference between the forces acting on the first and last portions, and dividing that difference by the number of terms diminished by unity. Now the force on the first portion is 404·5086, and that on the last is 50·5636 tons, and the number of terms or forces is 8; consequently we have $(404{\cdot}5086 - 50{\cdot}5636) \div 7 = 50{\cdot}5636$, which is the constant difference sought.

Therefore, if from the magnitude of the force which individually balances the weight, the constant difference be successively subtracted, the remainders will indicate the diminishing forces acting on the hypothenuse or chain; thus we have

$$b_0 = 404{\cdot}5086 - 50{\cdot}5636 = 353{\cdot}9450 \text{ tons.}$$

$$b_1 = 353{\cdot}9450 - 50{\cdot}5636 = 303{\cdot}3814 \text{ ,,}$$

$$b_2 = 303{\cdot}3814 - 50{\cdot}5636 = 252{\cdot}8178 \text{ ,,}$$

$$b_3 = 252{\cdot}8178 - 50{\cdot}5636 = 202{\cdot}2542 \text{ ,,}$$

$$b_4 = 202{\cdot}2542 - 50{\cdot}5636 = 151{\cdot}6906 \text{ ,,}$$

$$b_5 = 151{\cdot}6906 - 50{\cdot}5636 = 101{\cdot}1270 \text{ ,,}$$

$$b_6 = 101{\cdot}1270 - 50{\cdot}5636 = 50{\cdot}5634 \text{ ,,}$$

These forces, however, may also be deduced directly from the formula (K), instead of finding them as above from each other by continued subtraction; and since it may be necessary

on certain occasions to determine them in this way, it will be useful to show in what manner the reduction is to be effected.

By examining the several equations it will readily appear that $d \sin. \phi$ in each of them is constant, and is equivalent to $120 \sin. 18^\circ = 37{\cdot}08206$; and the several portions of the entire effect which belong to the respective forces $b_0$, $b_1$, $b_2$, $b_3$, $b_4$, $b_5$, and $b_6$, are 13125, 11250, 9375, 7500, 5625, 3750, and 1875; so that we have

$$b_0 = \frac{13125}{37{\cdot}08206} = 353{\cdot}9450 \text{ tons.}$$

$$b_1 = \frac{11250}{37{\cdot}08206} = 303{\cdot}3814 \quad ,,$$

$$b_2 = \frac{9375}{37{\cdot}08206} = 252{\cdot}8178 \quad ,,$$

$$b_3 = \frac{7500}{37{\cdot}08206} = 202{\cdot}2542 \quad ,,$$

$$b_4 = \frac{5625}{37{\cdot}08206} = 151{\cdot}6906 \quad ,,$$

$$b_5 = \frac{3750}{37{\cdot}08206} = 101{\cdot}1270 \quad ,,$$

$$b_6 = \frac{1875}{37{\cdot}08206} = 50{\cdot}5635 \quad ,,$$

If we suppose all the values of $\phi$ but the last, or $\phi_7$, to be equal to 90° or a right angle whose sine is unity; that is, if we suppose the subsidiary forces to act in directions perpendicular to the platform or lever, then it is obvious that the actual values of those subsidiary forces will be less than when they act in oblique directions; and for this reason, the several forces which act in the direction of the hypothenuse must be proportionally augmented in order to produce the same effect at the point of suspension; hence the advantage of causing the subsidiary forces to act obliquely; and the same principle applies to the oblique suspending rods in Mr. Dredge's method of constructing bridges.

# CONTRACT

FOR ERECTING

## A SUSPENSION BRIDGE UPON MR. DREDGE'S PRINCIPLE AT BALLOCH FERRY, LOCH LOMOND.

(See Plate 87.)

It is contracted and agreed between the parties underwritten, viz., Sir James Colquhoun, of Luss, Bart., in the county of Dunbarton, North Britain, on the first part; and James Dredge, of Bath, in the county of Somerset, England, engineer, &c., as principal; and William Gibbons, of Bath aforesaid, maltster, &c., as guarantee, cautioner, and surety on the second part, in manner following: that is to say,—Whereas the said Sir James Colquhoun has resolved to erect a suspension bridge at Balloch Ferry, in the parish of Bonhill, Dunbartonshire, for the transit of foot passengers, horses, carriages, and cattle, conform to plans, elevation, and measurement, designed, drawn, and subscribed by the said James Dredge upon the principle invented by him, which plans consist of Nos. 1, 2, 3, and 4 different parts, which are all subscribed by the said parties as relative to these presents. Therefore, and in consideration of the price hereinafter mentioned, the said James Dredge has bound and obliged himself, and by these presents binds and obliges himself, and his heirs, executors, and representatives whomsoever, to construct and finish the said bridge, of sufficient strength, and in a good and workmanlike manner, conform to the plans, elevation, and measurement above mentioned; subject always to such alterations as shall appear to the said James Dredge to

be rendered necessary for the proper strength and durability of said erection.

And the said James Dredge obliges himself, at his own proper costs and charges, to provide the whole requisite working drawings and sections, models for masonry and iron-work, and also to furnish all iron rails and timber, and all other materials whatsoever (masonry materials excepted), which shall be necessary and fit to be used in or about the said bridge, and to superintend the erection of the same, and of the whole mason-work therewith connected.

And more particularly the said James Dredge binds and obliges himself, and his foresaids, to erect and finish the said bridge in manner following, viz.: of twenty feet width, and two hundred feet in length from the centre of one tower to the centre of the other, and of not less than seventeen feet and one half foot in height above the level of the River Leven at high flood, or at least as high from the surface of the river as the Bonhill Bridge, with an opening of forty feet outside of each tower. The platform of the bridge to be three inches thick, of larch timber, and the other parts of good iron and good workmanship, with iron gates, and the whole of the iron-work to be once well painted; and in general, without prejudice to the particulars above specified, the said James Dredge binds and obliges himself, and his foresaids, to construct and finish the said bridge and openings conform to the foresaid plans, elevation, and measurement, with such alterations as may be rendered necessary as aforesaid; and that the whole of the said bridge shall be executed in the best and strongest manner, and of the best materials.

And the said second parties hereby warrant that the said bridge, when completed, shall be capable of sustaining in transit double the weight of any load, whether of cattle, carriages, or otherwise, that it may ever be fairly exposed to. And further, the said James Dredge binds and obliges himself, and his foresaids, to proceed to the execution of the work above mentioned, and to carry on the same without any intermission, and completely finish the whole within two calendar months immediately after the completion of the towers of the said bridge. Providing hereby and declaring, that if the said

James Dredge shall fail so to complete the said bridge within said period of two months, he shall be bound to allow to the said Sir James Colquhoun, and the latter shall be entitled to retain from the price after specified, an abatement of five per cent. upon the amount which the said Sir James Colquhoun shall have advanced for masonry on the said bridge, and that during the time the said bridge shall remain unfinished after the expiring of the foresaid two months.

And it is hereby agreed that if the fencing off of a foot-path upon said bridge shall be dispensed with by the said Sir James Colquhoun, he shall in that case be allowed a deduction of twenty pounds from the stipulated price of said bridge hereinafter mentioned; and in the event of his requiring only two gates on said bridge in place of three shown on said plan, he shall receive a further deduction on that account of five pounds sterling. And further, if any other part of the said plans shall be dispensed with by the said Sir James Colquhoun, or not executed by the said James Dredge, the expense of such part or parts shall also be deducted from the price after mentioned. And if any of the conditions or particulars above specified shall be altered or executed contrary to the foresaid plans, these shall not be construed into an abandonment of the said plans, but the same shall be completed, and the present contract shall be effectual notwithstanding such alterations.

And further, the said second parties, viz., the said James Dredge, as principal, and the said William Gibbons, as cautioner, guarantee, and surety for him as aforesaid, bind and oblige themselves, conjunctly and severally, and their foresaids, to warrant, uphold, and maintain the said bridge and towers thereof in good and sufficient condition for the space of five years after completion thereof, excepting from this obligation any injury or damage which shall arise thereto by lightning, earthquake, civil commotion, or malicious damage, or not fairly within the compass of wear and tear by ordinary use of such bridge. And if during the said period of five years any parts of said towers or bridge be discovered to be insufficient or requiring repair (not rendered necessary from the causes aforesaid), then the second parties shall be bound and

obliged, as they do hereby bind and oblige themselves and their foresaids, to repair such parts and make them sufficient at their own expense immediately on the insufficiency being discovered. For which causes, and on the other part, the said Sir James Colquhoun binds and obliges himself, his heirs and successors, immediately on such bridge being completed and proved to be sufficiently strong in all its parts as after mentioned, which proof shall be ascertained within one calendar month after its completion, to make payment to the said James Dredge, or his heirs, executors, or assigns, of the sum of one thousand five hundred pounds sterling, as the agreed price of the said bridge to be constructed by the said James Dredge in terms hereof, including therein the whole of his charges for engineering, travelling, and all other charges and expenses competent to him for said work, excepting such reasonable charge for superintending the erection of the said towers, either by himself or by a qualified person appointed by him, as may be considered fair between the parties, or as shall be fixed by reference to a respectable tradesman resident in Dunbartonshire, in case they do not agree. It being here contracted and agreed that the said Sir James Colquhoun shall, when required by the said James Dredge, furnish the timber requisite for said bridge, and be entitled to retain out of the said price of one thousand five hundred pounds the current price of such timber so furnished, besides the expense of dragging, conveyance, and whole workmanship thereof; and on the completion of said bridge, the price of said timber and the said expenses attending the same (which last shall be ascertained and proved by the account and statement of the wood forester on the estate of Luss) shall be deducted from the said stipulated sum of one thousand five hundred pounds; and the balance thereof, after such deduction, shall then be immediately payable to the said James Dredge and his foresaids. Providing always that the said bridge, including mason-work, shall have been previously duly tested and proved, within the aforesaid period of one calendar month, to be of the requisite strength, and completed in all respects according to agreement, and in such a manner as to afford to the public a safe com-

munication across the river Leven. But it is hereby expressly provided and declared that, if between the completion of the said bridge and the expiration of said five years, the bridge, including towers, shall not have fairly borne its work, and be thus proved unable or insufficient to answer the purposes for which the bridge is intended, the said James Dredge, as principal, and the said William Gibbons, as guarantee, cautioner, and surety foresaid, shall be bound and obliged, as they hereby bind and oblige themselves and their aforesaids, conjunctly and severally, to repair and make the bridge substantial in every respect, or to make repayment to the said Sir James Colquhoun, his heirs and assigns, not only of the foresaid sum of one thousand five hundred pounds sterling, but also of the further sum of two hundred pounds as and for a moiety of the outlay and expense disbursed by the said Sir James Colquhoun in the erection of the mason-work connected with the said bridge; and on such repayment, the said James Dredge and his foresaids shall be fully entitled to the whole of the iron and wood-work of the said bridge as his own property, and all the materials excepting the stone and mason-work thereof. And it is further hereby provided that, in the event of any difference arising with respect to the true meaning of the present contract or the execution of any part of the work hereby contracted for, the parties hereby submit the same to the final determination of William Steele, Esq., Advocate, Sheriff Substitute of Dunbartonshire, and failing him, to the Sheriff Substitute of the said shire for the time being, as sole arbiter, and oblige themselves and their foresaids to abide by and fulfil any decision which he shall pronounce on the matters hereby submitted to him. And both parties bind and oblige themselves and their foresaids to implement and perform their respective parts of the premises to each other under the penalty of five hundred pounds sterling, to be paid by the party failing to the party observing the contract or willing to do so, besides performance. And both parties consent to the registration hereof in the Books of Council and Session Sheriff Court of Dunbartonshire or other Judges' Books competent for preservation; and that letters of horning, or six days' charge, and all other

execution needful, may hereon pass on a decree to be interponed hereto, and for that purpose constitute

Procurators; in witness whereof these presents, written upon stamped paper by James Mackibbin, Clerk to Robert Grieve, Writer, Dunbarton, are subscribed,

(L. S.)
(L. S.)
(L. S.)

# SPECIFICATION OF THE QUANTITIES OF MATERIAL

USED IN THE

# SUSPENSION BRIDGE AT BALLOCH FERRY,

DUNBARTONSHIRE.

---

PLATE 87 is an isometrical projection of a suspension bridge now in course of erection across the Leven at Balloch Ferry in Dunbartonshire, for Sir James Colquhoun, of Luss, Bart., after designs and under the superintendence of Mr. Dredge, of Bath.

The whole extent of the suspended road-way is 292 feet; but as a space is left on each side as an easy access to and from the towing-paths, the distance from the centre of one pier to the centre of the other is reduced to 200 feet, which must be considered as the span of the bridge.

Each pier for supporting the chains consists of two octagonal towers upon which the chains rest, and which are connected together by springing a light arch across the road-way. These towers taper from 15 feet 2 inches by 9 feet 2 inches at the base, to 9 feet by 3 feet at the top, the longest dimension being in the direction of the bridge. The entire height of the towers is as under, viz.

| | feet. |
|---|---|
| From bed of river to water-mark . . . . . | 3 |
| ,, water-mark to upper surface of road-way . | 16 |
| ,, road-way to tops of towers . . . . . | 21 |
| | 40 feet. |

The distance from the centre of one octagonal tower to the centre of the other across the road-way is 20 feet, equal to the width of the bridge. The towers and the key-stone of the arch are built of a reddish freestone from Bonhill; the stringing courses at the road-way, springing of the arch, and the offsets at the top of the tower, of a white stone in the neighbourhood.

Upon the tops of the towers are large cast-iron plates, on which the chains rest.

The principal chains for supporting the structure are formed of $\frac{7}{8}$ round bars, laid side by side to the number of 13 upon the tops of the towers, and successively dropping one at each joint until it arrives at the centre, where that part of the chain is reduced to a single bar. The links upon the towers are 6 feet long, but after springing from thence they are increased to 9 feet.

1. The weight of each link, 6 feet long, is 14·5 lbs.
The weight of iron on the towers is therefore
$14{\cdot}5\,(13\times4)$ . . . . . $= 754$
The weight of each link, 9 feet long, is 20 lbs.
The weight of iron is therefore $20\{8(12+2\times5{\cdot}5)+4\} = 12400$
$(24\times4) = 96$ connecting bolts increasing in arithmetical progression from 3 to 12 lbs. each,
$\frac{96}{2}\,(3+12)$ . . . . . $= 720$

Weight of iron in chains . . $= 13874$ lbs.

The length of the oblique suspending rods, if radius remained constant, would be as the sec. of the angle which they respectively make with the horizon; but in consequence of the inclination of the chains towards the joints of connexion with the suspending rods, their length diminishes continually towards the centre of the bridge; and as the ascertaining the exact length of each rod separately would involve much complicated calculation, we will avoid it, and assume (which will be very near the truth) the average length of the bars to be 11 feet 9 inches, (excepting those immediately proceeding from

:he joints on the towers:) the two bars from each joint of the :hains being $\frac{3}{4}$ of an inch in diameter, we have, for the weight )f the whole of the $\frac{3}{4}$ bars,

| | | |
|---|---|---|
| ). | *1·47 (88 × 2 × 11·75) . . . . | = 3039·96 |
| | 16 oblique rods from the towers $\frac{7}{8}$ diameter, averaging 26·6 feet in length: | |
| | *2 (16 × 26·6) . . . . . | = 851·2 |
| | 136 pieces $\frac{7}{8}$ rod, each 2 feet long, welded to the end of the oblique rods: | |
| | 2 (136 × 2) . . . . . | = 544· |
| | 136 nuts, each ·25 lb. | |
| | ·25 × 136 . . . . . | = 34· |
| | 56 pieces 1 inch in diameter, 2 feet long, in retaining rods: | |
| | 2·61 (56 × 2) . . . . . | = 292·32 |
| | 56 nuts, each ·333 lb. | |
| | ·333 × 56 . . . . . | = 18·64 |
| | 152 nuts and pins, each ·5 lb. | |
| | ·5 × 152 . . . . . | = 76· |
| | Total weight of oblique suspending rods | = 4856·12 lbs. |

Proceeding onwards, we next arrive at the horizontal beams, )r those running from end to end of the bridge; and there are ıere 2 of 190 feet long, and 4 of 45 feet long each, all 5 inches × $\frac{1}{2}$ inch, so that

| | | |
|---|---|---|
| ). | 8·4 {(190 × 2) + (45 × 4)} . . . | = 4704 |
| | 64 connecting plates, each 1 foot long: | |
| | 8·4 × 64 . . . . . | = 537·6 |
| | 6 $\frac{5}{8}$ pins to each plate, each ·5 lb. | |
| | ·5 × 64 × 6 . . . . . | = 192 |
| | Total weight of horizontal beams | 5433·6 lbs. |

We have now to examine the weight of the transverse beams, r those which run at right angles to and immediately support

* Where 1·47 and 2 are respectively the weight in lbs. of a $\frac{3}{4}$ and $\frac{7}{8}$ rod 1 foot long.

the platform; they are each 20 feet long, 5 inches deep × ½ an inch in thickness, and have at every 6 inches along their length, near the upper edge, holes of $\frac{1}{10}$ths of an inch drilled to admit of $\frac{5}{8}$ rods passing longitudinally through them from end to end of the bridge; there are 117 such beams in all:

4. 8·4 (20×117) . . . . . = 19656

Every third beam is trussed with suspension trusses, consisting of 5 rods, viz., 1=$\frac{7}{8}$ and 4=$\frac{5}{8}$, each 6·66 feet long:

2 (39×6·66) . . . = 519·48

1·02 (39×4×6·66) . . =1059·739

1579·219

$\frac{1579·219 \times 8}{100}$ . . . = 126·337

1705·556= 1705·55

21361·55

From which deduct for the quantity of iron not used in the holes,

$1·23\left(\frac{40 \times 117}{24}\right)$ . . . . = 239·85

21121·70

The $\frac{5}{8}$ rods to which we before alluded run 6 inches apart through the transverse beams from end to end, and are placed there for the purpose of combining the whole mass together, and causing a more equable pressure upon the beams which support the road-way; there are, therefore, 40 rods of 195 feet long each, and 80 of 45 feet, so that the whole weight will be

5. 1·02 {(195×40) + (80×45)}=11628.

There are also 28 bars, each 10 feet long, $1\frac{1}{8}$ in diameter, to be used as holdfasts in the ground for retaining the chains.

6. 3·31 (28×10) . . . . . = 926·8

28 nuts, ·75 lb.

·75×28 . . . . . = 21

947·8 lbs.

And by adding the results together we shall have

| | | |
|---|---|---|
| 1 | . . | = 13874 |
| 2 | . . | = 4856·12 |
| 3 | . . | = 5433·6 |
| 4 | . . | = 21121·7 |
| 5 | . . | = 11628 |
| 6 | . . | = 947·8 |

| | | tons. | cwt. | qr. | lbs. |
|---|---|---|---|---|---|
| Total | 57861·22 | = 25 | 16 | 2 | 13·22 |

the whole of the wrought-iron actively employed in the structure; to which, if we add that used in railing,

| | |
|---|---|
| 2 (400 × 2 × 4) . . . . . | = 6400 |
| and about 400 lbs. more as staples, &c. . . | 400 |
| | 6800 lbs. |

| tons. | cwt. | qr. | lbs. |
|---|---|---|---|
| 3 | 0 | 2 | 24 |
| 25 | 16 | 2 | 13·22 |
| 28 | 17 | 1 | 9·22 |

the total quantity of wrought-iron used in the bridge.

We have next to examine the cast-iron used; and beginning with the plates on which the chains rest,

1. Four plates, each 560 lbs. . = 2240

136 circular plates for fastening the oblique rods to the horizontal beam, each 3·5 lbs.

2. 136 × 3·5 . . . . = 476

22 lbs. cast blocks to each truss, and there being 39 trusses:

22 × 39 . . . . = 858

2 cast boxes for each beam for attaching it to the horizontal longitudinal beams, each 11 lbs.

11 × 2 × 117 . . . = 2574

28 castings in retaining chains, each 5 lbs.

5 × 28 . . . . = 140

| | tons | cwt. | qr. | lbs. |
|---|---|---|---|---|
| 6288 lbs. = | 2 | 16 | 0 | 16 |

as the total quantity of cast-iron actively employed in the bridge; to which add the quantity used in the railing,

| | tons | cwt. | qr. | lbs. |
|---|---|---|---|---|
| $28 \times 4 \times 35 = 3920 =$ | 1 | 15 | 0 | 0 |
| | 2 | 16 | 0 | 16 |
| | 4 | 11 | 0 | 16 |

The timber used in its construction is larch, grown on the estate of Sir James Colquhoun at Luss, and the quantity used is,

$292 \times 20 \times \cdot 25 \quad . \quad . \quad . \quad . \quad . \quad = 1460$

2 beams of wood passing from end to end, $6 \times 3$ inches:

$2\,(292 \times \cdot 5 \times \cdot 25) \quad . \quad . \quad . \quad . \quad . \quad = \quad 73$

1533 feet.

The whole of the material used is,

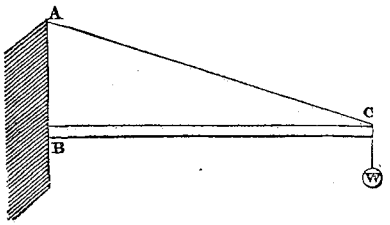
| | tons | cwt. | qr. | lbs. |
|---|---|---|---|---|
| 1st, Wrought-iron . . . . | 28 | 17 | 1 | 9·22 |
| 2nd, Cast-iron . . . . . | 4 | 11 | 0 | 16 |
| 3rd, Timber . . . . . | 1533 cubic feet. | | | |

The principle upon which this bridge is built is entirely new; and it has been doubted by many whether the arrangement of the bars in the chain and the adoption of the oblique suspending rods is a judicious application of iron in the construction of such bridges. The small quantity of material used in the one specified above, (which is intended as a permanent structure for general traffic,) is however, a sufficient proof; and we shall endeavour to show by mathematical reasoning, that it is more than equal to sustain any load which it is possible to place upon the surface of its road-way.

In calculating the force of a bridge of this description, the formulæ involved are necessarily very different from those used in estimating the common catenary; for here one half of the bridge balances the other, the structure being held in equilibrio entirely by the acting chains, and the centre part of the chain where it is reduced to a single link, may be severed or entirely removed without affecting the stability of the structure. It is therefore evident that such a bridge may be

referred to the projection of two brackets, and estimated accordingly.

Fig. 1.




Conceive the weight W to be supported by a cord A C at the extremity of a horizontal projection B C, and maintained there by a tension in the line A C, and compression in the line B C; then A B C being a right angled triangle by construction,

$$\text{sin. A C B} : \text{rad.} : : \text{A B} : \text{A C},$$

$$\text{or A C} = \frac{\text{rad.}}{\text{sin. A C B}} \cdot \text{A B} = \text{A B. cosec. A C B}$$

by the principles of projection.

$$\text{A B} : (\text{A C}) = \text{A B. cosec. A C B} : : \text{W} : \text{W} \frac{\text{cosec. A C B} \cdot \text{A B}}{\text{A B}} =$$

W cosec. A C B.

Let $w$=W, the weight resting on the point C.

$\theta = <$ A C B, contained by the horizontal line and the line A C.

$x$=the tension in the line A C (caused by the weight W) in the direction C A.

$y$=pressure or thrust in the direction C B.

Then, by substituting these quantities for the former proportion, we have

$$\text{sin. } \theta : \text{rad.} : : w : x,$$

and by equating the products and reducing the equation, we get

$$x = \frac{\text{rad.}}{\text{sin. } \theta} w = w \text{ cosec. } \theta \quad . \quad . \quad . \quad . \quad . \quad (1)$$

And for the pressure or thrust in the direction C B, it is

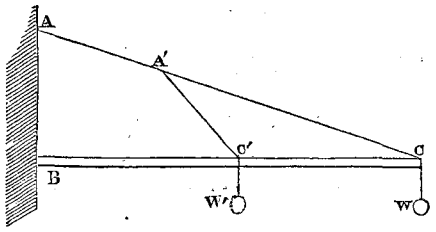
$$\text{sin. } \theta : \text{cos. } \theta : : w : y;$$

consequently by reduction, we have

$$y = \frac{\cos.\,\theta}{\sin.\,\theta} w = w \cot.\,\theta \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Which equation holds true for every value of the angle $\theta$; but there is the same strain in every part of the line A C, and if such were to remain, there would be no occasion to increase the size of the chain towards the point A. But let us further suppose another weight W′ to be supported by a line from some intermediate point between B and C, and let that line be attached to some point A′ in the cord A C, as in fig. 2.

Fig. 2.




Allowing A C B to remain as before, and supposing the same quantities to retain the same values as in the former figure, and putting in this $w'$, $\theta'$, $x'$, $y'$ respectively for the weight W′, the $<$ A′ C′ B, the tension in the line A′ C′, and the pressure in the direction C′ B, we have, for the value of $x'$ by equation (1),

$$x' = w' \text{ cosec. } \theta' \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The point A′ in the line A C is acted upon by two forces represented by the symbols $x$, $x'$, the resultant of which forces in the direction towards A will be,

$$R = \sqrt{x^2 + x'^2 + 2\,x\,x' \cos.\,(180° + \theta - \theta')} \quad . \quad . \quad (4)$$

The former values of $x$ and $x'$ being substituted, we get

$$R = \sqrt{w^2 \text{cosec.}^2\theta + w'^2 \text{cosec.}^2\theta' + 2w \text{cosec.}\,\theta \,.\, w' \cos.\,\theta' \cos.\,(180° + \theta - \theta')}$$

And the difference will be represented by

$$D = \sqrt{w^2 \text{cosec.}^2\theta + w'^2 \text{cosec.}^2\theta' + 2w \text{cosec.}\,\theta \,.\, w' \cos.\,\theta' \cos.\,(180° + \theta - \theta')} - w \text{ cosec. } \theta,$$

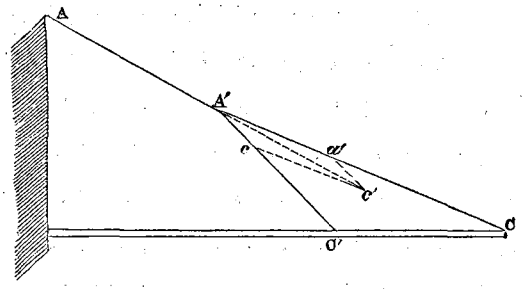
and in which proportion, the chains should increase in sectional area towards the base. Proceeding in like manner for another section, the resultant of the forces would be

$$R'' = \sqrt{(x''^2 + R^2 + 2\,R\,x'' \cos.\,(180° + \delta - \theta'')}* \qquad (5)$$

From which it appears, that each progressive section of the chain towards the point of suspension, has an increase of strain, equal to the resultant of the two forces acting at the further extremity of that section towards the extremity of the bracket; and from this it is evident, that the chain should taper in the same proportion; but it is equally clear, that the use of the oblique suspending rods is peremptorily insisted upon, otherwise the reduction of material towards the extremity of the bracket beyond a certain extent would be a proportionate reduction of power.

The following diagrams will graphically illustrate this:

Fig. 3.

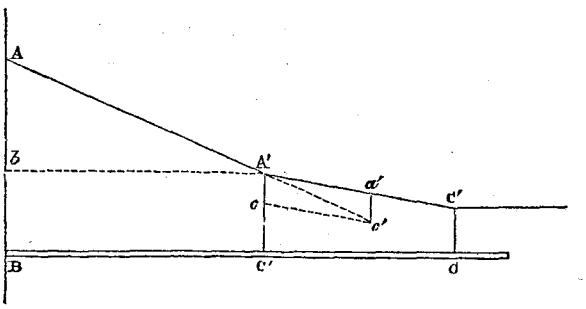



Draw $a'\,c'$ (fig. 3) parallel to A′ C′, produce A A′ onwards to the point $c'$, and the triangle of forces A′ $a'$ $c'$ being complete, it only remains that the lines supporting the brackets should be increased in power proportionally to the length of the sides of the triangle to which they are respectively parallel.

---

* Where $x''$ represents the value of the tension sustained in the angle or inclination $\theta''$, R the resultant of the first two forces, and $\delta$ an angle of which R is equivalent to the cosec.

Fig. 4.*




Again (fig. 4), draw $a'$ $c'$ parallel to A′ C′, and continue A A′ to $c'$; then will the triangle of forces be complete, as in the former curve; but it is evident by the construction of the bracket, that the pressure of the horizontal force is not sustained by the road-way. Draw A′ $b$ parallel to B C, then will A′ $b$ be the measure of that horizontal force, or equal to the weight resting on A′ drawn into the cotangent of the angle A A′ $b$, which must be counteracted by a sufficient power in the direction A′ C′; and the weight of the iron, which would be equivalent to that power, would be oppressing the chain in the direction A′ A, as the cosecant of the angle A A′ $b$.

It is well known that the parameter of the catenary of a suspension bridge, is equivalent to half the weight of the structure drawn into the cotangent of the angle which a tangent to the curve at the point of suspension forms with the horizon, and the oblique strain is equal to half the weight drawn into the cosecant of the same angle.

Therefore, from what has been said, it is clear that a parallel section of iron, equivalent to the parameter of the catenary, may be taken away from one end of the curve to the other, and that not only the weight would be saved, but also another

* I would here presume that whilst speaking of a bracket supported by a catenary curve with vertical suspending rods, a tensile force must be acting from the point C′ equivalent to the parameter of that curve.

quantity of iron which was necessary to support it. These, then, are the advantages which arise from the use of the oblique suspending rod.

*Example.*

Suppose A B C (fig. 1) to be a bracket projecting from a wall with a weight of 364 lbs. attached at the point C, the angle A C B being 39°; required the tensile power upon the cord A C, and also the horizontal pressure in the direction B C?

By equation (1) we have,

$$x = w \text{ cosec. } \theta,$$

therefore,

$$364 \times 1{\cdot}589015 \ . \ . \ . \ . \ . = 578{\cdot}401 \text{ lbs.},$$

and by logarithms,

| | | |
|---|---|---|
| log. 364 . . | = 2·561101 | |
| log. cosec. 39° | = 10·201128 | |
| nat. num. | 2·762229 | = 578·401 lbs. |

As shown by equation (2),

$$y = w \text{ cot. } \theta,$$

hence

$$364 \times 1{\cdot}234897 \ . \ . \ . \ . \ . = 449{\cdot}502 \text{ lbs.},$$

by logarithms,

| | | |
|---|---|---|
| log. 364 . . | = 2·561101 | |
| log. cot. 39° . | = 10·091631 | |
| nat. num. | 2·652732 | = 449·502 lbs. |

Again, let every thing remain as before; but suppose another weight equal to 364 lbs. were added to the point C′, as in fig. 2, and supported at an angle of 50°; required the increase of strength from A′ towards A, and also the horizontal pressure towards B?

By equation (3),

$$x' = w' \text{ cosec. } \theta';$$

consequently,

$$\begin{aligned} &364 \times 1{\cdot}305408 \quad . \quad . \quad . \quad . \quad . \quad = 475{\cdot}16 \text{ lbs.} \\ &\log.\ 364 \quad . \quad . \quad . \quad = 2{\cdot}561101 \\ &\log.\ \text{cosec.}\ 50^\circ \quad . \quad = 10{\cdot}115746 \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \\ &\qquad\qquad\qquad\qquad \text{nat. num.} \quad 2{\cdot}676847 \quad = 475{\cdot}16 \text{ lbs.} \end{aligned}$$

Therefore 475·16 lbs. is the tension in the direction C′ A′; but to ascertain the value of the tension from A′ to A, by referring to equation (4) we find,

$$R = \sqrt{x^2 + x'^2 + 2\,x\,x' \cos.\ (180^\circ + \theta - \theta')},$$

and by substituting the numerical values for the several quantities, we get

$$R = \sqrt{578{\cdot}401^2 + 475{\cdot}16^2 + 2 \times 578{\cdot}401 \times 475{\cdot}16 \times {\cdot}981627}$$

that is

$$\begin{aligned} 578{\cdot}401^2 &= 334547{\cdot}7168 \\ 475{\cdot}16^2 &= 225777{\cdot}0256 \\ 2 \times 578{\cdot}401 \times 475{\cdot}16 \times 981627 &= 539567{\cdot}0241 \end{aligned}$$

and

$$\sqrt{1099891{\cdot}7665} = 1048{\cdot}75 \text{ lbs.}$$

The required strength in that part of the cord A A′, and the difference between the part A A′ and A′ C is,

$$\begin{array}{r} 1048{\cdot}75 \\ 578{\cdot}401 \\ \hline 470{\cdot}349 \end{array}$$

And the sectional area of the one part should be to the sectional area of the other as

$$1 : 1{\cdot}8131.$$

For the magnitude of the horizontal force; that which is occasioned by the weight at the further extremity of the bracket has been shown to be = 449·502 lbs.; but the adding of another weight at C′ will cause an increase towards B, so that the aggregate of force from C′ to B is represented by

$$449{\cdot}502 + (364 \times {\cdot}839100) = 754{\cdot}934 \text{ lbs.}$$

And the difference of pressure between the part C′ B and C C′ is

$$754{\cdot}934 - 449{\cdot}502 = 305{\cdot}432.$$

*Example* 2.

Suppose a figure similar in every respect to fig. 4, and let a weight of 364 lbs. be attached to the horizontal line from the point C, and let A′ C′ form the same angle with the horizon that A′ C′ did in the former curve, viz., 39.° Then the tension in the direction A′ C′ will be

$$364 \times 1{\cdot}589015 = 578{\cdot}400 \text{ lbs.}$$

and the horizontal force, as before represented, is equal to

$$364 \times 1{\cdot}234897 = 449{\cdot}502 \text{ lbs.},$$

which must in this case be resisted through the cord in the direction from C; or, in short, as the parameter of the catenary; and therefore, the absolute tension in the direction C′ A is in consequence, equal to 1·589015 (364 + the weight of material requisite to sustain a pull of 449·502 lbs.), and which, in large spans, increases to such an enormous extent as to facilitate the destruction of the structure.

The throwing the thrust on the towers or abutments into the horizontal line is of advantage to the road-way, as rendering it more steady and not so liable to that oscillation to which suspension bridges are commonly exposed; and in practice, the sectional area of the beam to sustain the transverse action of the platform, would be more than sufficient under any circumstance, to overcome the horizontal action to which the structure may be exposed: for that reason, therefore, it would be useless to increase the beam towards B.

The preceding examples which refer to the bracket only are strictly arbitrary, and have been proposed for the purpose of showing in what manner the formulæ arising from the investigation are to be applied; but as a further illustration of the advantages to be derived from the adoption of the system just

discussed, we shall revert to the bridge across the River Leven for which the quantity of materials has already been ascertained. The whole quantity of material employed is apparently very small, and if compared with that which would be required on the old principle, might tend to create a doubt as regards the strength and stability of the structure. In order, therefore, to remove any suspicions that may be entertained on that point, we shall here endeavour to show that the material used, though small in quantity, is amply sufficient to sustain a much greater load, than under any possible circumstances can ever be brought upon the platform.

There are four points of suspension, and at each point there are twelve bars in the chains besides two oblique suspending rods, all of $\frac{7}{8}$ths of an inch in diameter; this gives a section of 8·4185 square inches at each tower or point of support, or, in all, 33·674 square inches of section. Now, since the absolute tensile power of a $\frac{7}{8}$th bar of malleable iron of a medium quality is 35000 lbs. very nearly, the absolute strength of the iron at the joints adjacent to the towers is 875 tons, for

$$(12=2)\times 4\times 35000=1960000 \text{ lbs.}=875 \text{ tons.}$$

In the next place, we have to ascertain the greatest tension, that under any circumstances can come upon the chains at the points of support, and for this purpose we shall suppose the weight of the platform, chains, diagonal rods, &c., between the towers to be 67200 lbs., and taking the greatest extraneous load to which a bridge ought ever to be exposed at 120 lbs. per square foot, we get

$$190\times 20\times 120=456000 \text{ lbs.}$$

the length of the platform being 190 feet; therefore we have

$$67200+456000=523200 \text{ lbs.}$$

for the total weight to be supported.

Now, the angle which the first series of bars would make with the platform, if continued downwards till they intersect it, is 24 degrees, and the natural cosecant of 24 degrees, is 2·458591; consequently, by equation 1 (page lv), we obtain for the effect of obliquity,

$$523200 \times 2{\cdot}458591 = 1286334{\cdot}8112 \text{ lbs., or } 574{\cdot}2566 \text{ tons.}$$

Let this be subtracted from the absolute power of the iron at the points of suspension, and we have

$$875 - 574{\cdot}2566 = 300{\cdot}7434 \text{ tons,}$$

which is the surplus strength in the chains after the platform is fully loaded; and every part of the structure, if properly tested, would be found equally sufficient for the intended purpose.

The preceding method of obtaining the surplus strength of the bridge is simple and easily applied, but it may be done otherwise as follows.

We have found that the entire weight to be supported in an extreme case is 523200 lbs., and we have moreover seen, that the ultimate strength of a single bar of the iron used, is 35000 lbs.; consequently, the number of bars required at the towers will be

$$523200 \div 35000 = 14{\cdot}948 \text{ bars.}$$

This, however, is on the supposition that the iron is strained vertically in the direction of its fibres, but this in the case of a bridge is contrary to fact, the bars being all placed in an oblique position with respect to the horizon, while the weight acts in a direction perpendicular to it; the power of the iron must therefore be reduced in the ratio of radius to the sine of the angle of obliquity; or, which is the same thing, the weight must be increased in the ratio of the cosecant to radius. Now the angle of obliquity is 24 degrees, and its cosecant 2·458591; hence we get

$$523200 \times 2{\cdot}458591 \div 35000 = 36{\cdot}75 \text{ bars;}$$

but there are 56 bars at the towers in the Balloch Bridge; consequently we have

$$56 - 36{\cdot}75 = 19{\cdot}25 \text{ bars,}$$

or 4·8125 bars at each tower for the surplus strength, even in the case of an extreme load.